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On Spherical Product Surfaces in E^3

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***Abstract*—In the present study we consider spherical product surfaces $X = \alpha \otimes \beta$ of two 2D curves in E^3. We prove that if a spherical product surface patch $X = \alpha \otimes \beta$ has vanishing Gaussian curvature K (i.e. a flat surface) then either α or β is a straight line. Further, we prove that if $\alpha(u)$ is a straight line and $\beta(v)$ is a $2D$ curve then the spherical product is a non-minimal and flat surface. We also prove that if $\beta(v)$ is a straight line passing through origin and $\alpha(u)$ is any $2D$ curve (which is not a line) then the spherical product is both minimal and flat. We also give some examples of spherical product surface patches with potential applications to visual cyberworlds.**

***Keywords*-spherical product surface; minimal surfaces; function based geometry modelling;**

I. Introduction

The problem of constructing geometry of objects which resemble real world objects is important in many areas of computer graphics and computer vision. These include robotics, medical image analysis and the automatic construction of virtual environments. In the last 30 years, much effort has been focussed in finding suitable methods representing objects from 3D data. This work has largely proposed the use of some form of parametric models, most commonly spherical product of two 2D curves.

Quadrics are the simplest type of spherical products. In fact, the first dedicated part-level models in computer vision were generalized cylinders [3]. Superquadrics can be also considered as spherical product of two $2D$ curves which are known as superconics. In fact, superquadrics are solid models that posses fairly simple parametrization and can represent a large variety of standard geometric solids, as well as smooth shapes in between. This makes them much more convenient for representing rounded, blob-like geometry which resemble common objects formed by natural processes [12].

Petland was first who grasped the potential of the superquadratic models and parametric deformations for modelling natural shapes in the context of computer vision [17]. He proposed to use superquadrics models, in combination with global deformations. This was proposed as a set of primitives which can be molded like clay which can be intuitive for the user. For example, Petland presented several perceptual and recognizable arguments to recover the scene structure at such a part-level. He proposed superquadrics in combination with deformations as a shape vocabulary for such part-level representation.

The superquadrics, which are like phonemes in this description can be deformed by stretching, bending, tapering or twisting and then can be combined using Boolean operations to build complex objects ([12], pp. 9). The study of superquadric model started in isolation from specific vision applications ([17], [4], [19]). It can be observed that superquadric recovery can be integrated with segmentation ([18], [11], [14]) as well as with decision making such as categorization [13]. Superquadrics are the special case of the supershapes, developed by Gielis and et al. [8] that have the advantage of representing polygonal geometry with various symmetries.

The rest of the paper is organized as follows. Section II provides a formal definition of spherical product surfaces and superquadrics with global parametrization. Section III presents the original results of spherical product surface patches of flat or minimal type and results of deformations of the spherical product surface patches. Section IV provides some examples and finally, Section V concludes the paper.

II. Spherical Product Surfaces in E^3

Let $\alpha, \beta : R \longrightarrow E^2$ be two Euclidean planar curves. Assume $\alpha(u) = (f_1(u), f_2(u))$ and $\beta(v) = (g_1(v), g_2(v))$. Then their spherical product immersion is given by,

$$X = \alpha \otimes \beta : E^2 \longrightarrow E^3; \qquad (1)$$
$$X(u,v) = (f_1(u), f_2(u)g_1(v), f_2(u)g_2(v)),$$

$u_0 < u < u_1, v_0 < v < v_1$, which is a surface in E^3 [12]. Each $2D$ curve has one degree of freedom, so the

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resultant surface has 2 degrees of freedom. By adding a scaling term to each spatial direction, we achieve a form of with 5 degrees of freedom,

$$X(u,v) = (a_1 f_1(u), a_2 f_2(u) g_1(v), a_3 f_2(u) g_2(v)). \quad (2)$$

We can think of the function β as horizontal curve which is swept vertically according to the function α. Further, $f_1(u)$ scales β while $f_2(u)$ defines the vertical sweeping motion. In this way we see that the parameter v attaches the surface horizontally, while u attaches the surface vertically [2]. For the case $\beta(v)$ is a unit circle one can get a parametrization of a surface of revolution,

$$X(u,v) = (f_1(u), f_2(u)\cos v, f_2(u)\sin v). \quad (3)$$

Quadratic surfaces occur frequently in the design of discrete piece parts in mechanical CAD/CAM. Solid modeling systems based on quadratic surfaces must be able to allow the underlying surface to be partitioned [16]. The quadratic surface can also be represented in an explicit way using spherical product of two 2D curves [12]. Some examples are listed in Table 1 ([15]).

A. *Superquadrics*

The circle and square, ellipse and rectangle are all members of the set of superellipses defined by,

$$\left|\frac{x_1}{a_1}\right|^{\frac{2}{\epsilon_2}} + \left|\frac{x_2}{a_2}\right|^{\frac{2}{\epsilon_2}} = 1, \quad (4)$$

where the lengths of the axes are given by a_1 and a_2 and the squareness is determined by ϵ [6]. Superellipse was developed as a popular tool by Piet Hein and has been used for shape design by architects and furniture designers [5]. The solutions of Equation (2) can be parameterized as,

$$\begin{bmatrix} x_1(v) \\ x_2(v) \end{bmatrix} = \begin{bmatrix} a_1 \cos^{\epsilon_2} v \\ a_2 \sin^{\epsilon_2} v \end{bmatrix} - \pi \leq v < \pi. \quad (5)$$

Superquadrics [12] are a family of parametric solids derived from the basic quadric surfaces and solids. Extra flexibility in shape representation is achieved by raising each trigonometric term in the quadric equations to an exponent. These exponents control the relative roundness and squareness along the major axes of the surface. By altering the value of the exponents, a wide range of forms may be generated. e.g. spheres, cylinders. parallelepipeds, pinched stars and the shapes in between. Superquadrics are a family of shapes that includes not only superellipsoids, but also superhyperboloids of one piece and superhyperboloids of two pieces as well as supertoroids.

In computer vision literature, it is common to refer to superellipsoids by the more generic term of superquadrics. The following position vector X defines a superquadric surface,

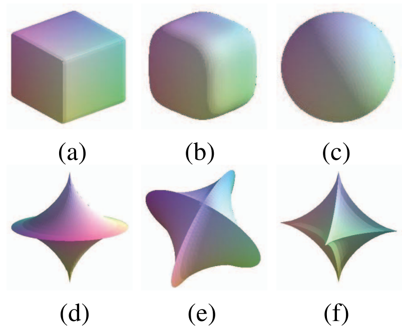

Figure 1. Superquadric shapes varying ϵ_1, ϵ_2. (a) $\epsilon_1 = \epsilon_2 = 0.1$, $(b)\epsilon_1 = \epsilon_2 = 0.5$, $(c)\epsilon_1 = \epsilon_2 = 1$, (d) $\epsilon_1 = 3, \epsilon_2 = 1$, $(e)\epsilon_1 = 1, \epsilon_2 = 3$, $(f)\epsilon_1 = \epsilon_2 = 3$.

$$\begin{aligned} X(u,v) &= \alpha(u) \otimes \beta(v) \quad (6) \\ &= \begin{bmatrix} a_1 \sin^{\epsilon_1} u \\ \cos^{\epsilon_1} u \end{bmatrix} \otimes \begin{bmatrix} a_2 \cos^{\epsilon_2} v \\ a_3 \sin^{\epsilon_2} v \end{bmatrix} \\ &= \begin{bmatrix} a_1 \sin^{\epsilon_1} u \\ a_2 \cos^{\epsilon_1} u \cos^{\epsilon_2} v \\ a_3 \cos^{\epsilon_1} u \sin^{\epsilon_2} v \end{bmatrix}, \end{aligned}$$

where $-\frac{\pi}{2} < u < \frac{\pi}{2}$ and $-\pi \leq v < \pi$.

Superquadric is a well-known part-level model in the field of computer vision and graphics. Being an extension of the quadric surfaces the superquadric incorporates two shapes control parameters ϵ_1 and ϵ_2 to adjust the curvature of the surface (see, [1], [12] and [12]). When ϵ_1, ϵ_2 vary, the shape smoothly changes. In the special case $\epsilon_1 = \epsilon_2 = 1$, the superquadric degenerates to a common ellipsoid (see, Figure 1).

By eliminating parameter u and v using equality $cos^2\alpha + sin^2\alpha = 1$, the following implicit equation,

$$\left(\left|\frac{x_2}{a_2}\right|^{\frac{2}{\epsilon_2}} + \left|\frac{x_3}{a_3}\right|^{\frac{2}{\epsilon_2}} \right)^{\frac{\epsilon_2}{\epsilon_1}} + \left|\frac{x_1}{a_1}\right|^{\frac{2}{\epsilon_1}} = 1. \quad (7)$$

can be obtained.

B. *Supershapes*

Supershapes have been recently presented by Gielis ([6],[8]) as an extension of superquadrics deriving from superellipse representation. Here a term $\frac{m\theta}{4}$, $m \in R^+$, is introduced to allow a rational or irrational number of symmetry and three shape coefficients are considered. The radius r of a polygon is defined by,

$$r(\theta) = \frac{1}{\left(\left|\frac{\cos(m\theta)}{a_1}\right|^{n_2} + \left|\frac{\sin(m\theta)}{a_2}\right|^{n_3} \right)^{\frac{1}{n_1}}}, \quad (8)$$

with a_1, a_2, $n_i \in R^+$ and $m \in R^+_*$. Parameters $a_1 > 0$ and $a_2 > 0$ controlling the size of the polygon, defines the number of symmetry axes and can also be seen as the number of sectors in which the plane is folded. When m is a natural number, non-self-intersecting closed curves are obtained. For $n_1 = n_2 = n_3 = 2$ and $m = 4$ in Equation

$\alpha(u)$	$\beta(v)$	$X(u,v) = \alpha \otimes \beta$
circle with radius r	circle radius r, $g_1(v) \geq 0$	sphere with radius r
circle	line $x = x_{const} > 0$	cylinder
circle	line	cone
circle	eclipse with $g_1(v) \geq 0$	rotation ellipsoid
ellipse, parabola or hyperbola	line $x = x_{const} > 0$	elliptic, parabolic or hyperbolic cylinder
ellipse	line	elliptic cone
ellipse	ellipse with $g_1(v) \geq 0$	ellipsoid
ellipse	ellipse with centre $x \geq a_g$	toroid
ellipse	one sheeted hyperbola	one sheeted hyperboloid
hyperbola	one sheeted hyperbola	two sheeted hyperboloid
ellipse or hyperbola	parabola[1] with $g_1(v) \geq 0$	elliptic or hyperbolic paraboloid

Table 1. Some quadrics defined as spherical products.

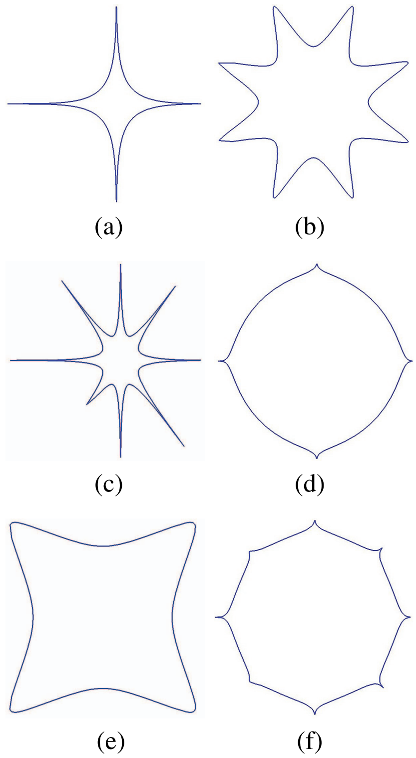

Figure 2. Examples of various abstract shapes. (a) and (c) $n_1 = n_2 = n_3 = \frac{1}{3}$, (b) and (e) $n_1 = 10, n_2 = n_3 = 20$, (d) and (f) $n_1 = 3, n_2 = n_3 = \frac{1}{3}$.

(8), an ellipse is obtained. One can find in nature a variety of interesting shapes that may possibly be described by the formula (8).

When combined with another function $f(\theta)$, the Superformula will modify these functions and all associated graphs (Eq. 9),

$$\rho(\theta) = \frac{f(\theta)}{\left(\left|\frac{\cos(m\theta)}{a}\right|^{n_2} + \left|\frac{\sin(m\theta)}{a}\right|^{n_3}\right)^{\frac{1}{n_1}}}. \tag{9}$$

The function $f(\theta)$ may be considered as a modifier of the Gielis function, $r(\theta)$ [9]. Functions $f(\theta)$ may be for example, constant functions ((8) = (9)), exponential functions, spiral functions and trigonometric functions ([6], [7]).

This generic equation generates a large class of supershapes and subshapes, including the supercircles and subcircles as special cases. Gielis therefore proposed the name Superformula for Equation (9) based on the notion of supercircles, superellipses and superquadrics.

Thus, supershapes have been recently presented by Gielis [6], [8] as an extension of superquadrics. A considered parametric equation of supershapes can be written as,

$$X(u,v) = \alpha(u) \otimes \beta(v) = \tag{10}$$
$$\begin{bmatrix} r_1(u)\sin u \\ r_1(u)\cos u \end{bmatrix} \otimes \begin{bmatrix} r_2(v)\cos v \\ r_2(v)\sin v \end{bmatrix}$$
$$= \begin{bmatrix} r_1(u)\sin u \\ r_1(u)r_2(v)\cos u\cos v \\ r_1(u)r_2(v)\cos u\sin v \end{bmatrix},$$

where $-\frac{\pi}{2} < u < \frac{\pi}{2}$ and $-\pi \leq v < \pi$.

A unit supershape ($a = b = 1$) is defined by 8 shape parameters denoted $\{m, n_1, n_2, n_3, \widetilde{m}, \widetilde{n}_1, \widetilde{n}_2, \widetilde{n}_3\}$, where n_i and $\widetilde{n}_i$ are used in $r_1(u)$ and $r_2(v)$ respectively [8].

III. Main Results

We recall definitions and results of [10].

Let $\alpha, \beta : R \longrightarrow E^2$ be two Euclidean planar curves. Assume $\alpha(u) = (f_1(u), f_2(u))$ and $\beta(v) = (g_1(v), g_2(v))$. Then their spherical product immersion is given by,

$$X = \alpha \otimes \beta : E^2 \longrightarrow E^3, \tag{11}$$
$$X(u,v) = (f_1(u), f_2(u)g_1(v), f_2(u)g_2(v)),$$

which is a surface in E^3.

The tangent space of $X(u,v)$ is spanned by the vector fields,

$$X_u(u,v) = (f_1{}'(u), f_2{}'(u)g_1(v), f_2{}'(u)g_2(v)), \tag{12}$$
$$X_v(u,v) = (0, f_2(u)g_1{}'(v), f_2(u)g_2{}'(v)), \tag{13}$$

where f' means the derivative of f.

Hence, the coefficients of the first fundamental form of the surface are,

$$E =< X_u(u,v), X_u(u,v) > \tag{14}$$
$$= (f_1{}'(u))^2 + (f_2{}'(u))^2 \|(\beta(v))\|^2$$
$$F =< X_u(u,v), X_v(u,v) > \tag{15}$$
$$= f_2(u) f_2{}'(u) < \beta(v), \beta'(v) >$$
$$G =< X_v(u,v), X_v(u,v) > \tag{16}$$
$$= (f_2(u))^2 \left\|\beta'(v)\right\|^2,$$

where $\langle , \rangle$ is the standard scalar product in E^3.

For a regular patch $X(u,v)$ the unit normal vector field or surface normal N is defined by,

$$N(u,v) = \frac{X_u \times X_v}{\| X_u \times X_v \|}(u,v), \tag{17}$$

where,

$$\|x_u \times x_v\| = \sqrt{EG - F^2}$$
$$= f_2 \sqrt{(f_1')^2 \{(g_1')^2 + (g_2')^2\} + (f_2')^2 \{g_1 g_2' - g_1' g_2\}^2},$$
$$f_2 \neq 0.$$

does not vanish [10].

The second partial derivatives of $X(u,v)$ are expressed as follows,

$$X_{uu}(u,v) = (f_1''(u), f_2''(u) g_1(v), f_2''(u) g_2(v)), \tag{18}$$
$$X_{uv}(u,v) = (0, f_2'(u) g_1'(v), f_2'(u) g_2'(v)), \tag{19}$$
$$X_{vv}(u,v) = (0, f_2(u) g_1''(v), f_2(u) g_2''(v)). \tag{20}$$

Similarly, the coefficients of the second fundamental form of the surface are,

$$e =< X_{uu}(u,v), N(u,v) >$$
$$= \frac{f_2(u)}{\sqrt{EG - F^2}} A(u) B(v), f_2(u) \neq 0,$$
$$f =< X_{uv}(u,v), N(u,v) >= 0,$$
$$g =< X_{vv}(u,v), N(u,v) >$$
$$= \frac{(f_2(u))^2 f_1{}'(u)}{\sqrt{EG - F^2}} C(v), f_2(u) \neq 0,$$

where,

$$A(u) = (f_1''(u) f_2'(u) - f_2''(u) f_1'(u)), \tag{21}$$
$$B(v) = (g_1(v) g_2'(v) - g_2(v) g_1'(v)), \tag{22}$$
$$C(v) = (g_2''(v) g_1'(v) - g_2'(v) g_1''(v)). \tag{23}$$

Furthermore, the Gaussian and mean curvatures of the surface becomes,

$$K = \frac{eg - f^2}{EG - F^2}$$
$$= \frac{(f_2(u))^3 f_1{}'(u)}{(EG - F^2)^2} A(u) B(v) C(v); f_2(u) \neq 0,$$

and

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}$$
$$= \frac{f_2{}^2 \left\{ f_1{}' [A_1] C(v) + f_2 \left\|\beta'(v)\right\|^2 A(u) B(v) \right\}}{2(EG - F^2)^{\frac{3}{2}}}$$

respectively. Here $A_1 = (f_1{}')^2 + (f_2{}')^2 \|(\beta(v))\|^2$.

Summing up the following results are proved.

Theorem 1: Let $X(u,v) = \alpha(u) \otimes \beta(u)$ be the spherical product surface patch of two planar curves. If $X(u,v)$ is a flat surface patch (i.e. $K = 0$) in E^3 then either $\alpha(u)$ (or $\beta(v)$) is a straight line, or $f_1{}'(u) = 0$.

Proof: Suppose the spherical product immersion $\alpha \otimes \beta$ of two planar curves is a flat surface. Then by Equation (24) one of the terms $f_1'(u)$, $A(u)$, $B(v)$, or $C(v)$ vanishes identically. For the case $f_1'(u) = 0$, the spherical product surface becomes a part of a plane. Furthermore, $A(u) = 0$ (or $C(v) = 0$) implies that $\alpha(u)$ (or $\beta(v)$) is a straight line. For the case $B(v) = 0$, the curve $\beta(v)$ is a straight line passing through the origin. This completes the proof of the theorem.

Theorem 2: The spherical product surface patch $X(u,v) = \alpha(u) \otimes \beta(u)$ of two planar curves α and β is minimal (i.e. $H = 0$) in E^3 if and only if,

$$f_1{}'(u) \left[(f_1'(u))^2 + (f_2{}'(u))^2 \|(\beta(v))\|^2 \right] C(v) \tag{24}$$
$$+ f_2 \left\|\beta'(v)\right\|^2 A(u) B(v) = 0.$$

Proof: Suppose the spherical product patch $X(u,v)$ of two planar curves is a minimal. Then by definition the mean curvature H vanishes identically. So, by the use of Equation (24) we get (25).

By using Theorem 1, we obtain the following.

Corollary 1: Let $X(u,v)$ be a spherical product surface patch of two $2D$ curves $\alpha(u) = (f_1(u), f_2(u))$ and $\beta(v) = (g_1(v), g_2(v))$.

i) If $\alpha(u)$ is a straight line with $f_1'(u) = 0$ then the surface becomes a part of a plane.

ii) If $\alpha(u)$ is a straight line with $f_2'(u) = 0$, then the surface becomes a cylinder over the curve $\beta(v)$.

iii) If $\alpha(u)$ is a straight line with $f_2(u) = m f_1(u) + n$ then the surface becomes conical.

By using Theorem 2, we obtain the following.

Corollary 2: Let $X(u,v)$ be a spherical product surface patch of two $2D$ curves $\alpha(u) = (f_1(u), f_2(u))$ and $\beta(v) = (g_1(v), g_2(v))$.

i) If $\alpha(u)$ is a straight line and $\beta(v)$ is a $2D$ curve (which is not a straight line) then the spherical product is a non-minimal and flat surface.

ii) If $\beta(v)$ is a straight line passing through origin and $\alpha(u)$ is any $2D$ curve then the spherical product is both minimal and flat.

iii) If $\alpha(u)$ is the 2D curve Catenary given with the parametrization $\alpha(u) = (u, a\cosh(\frac{u}{a} + b))$, $a, b \in R$, $a \neq 0$, and $\beta(v)$ is a unit circle then the surface patch $X(u,v)$ is a surface of revolution which is minimal and non-flat [10].

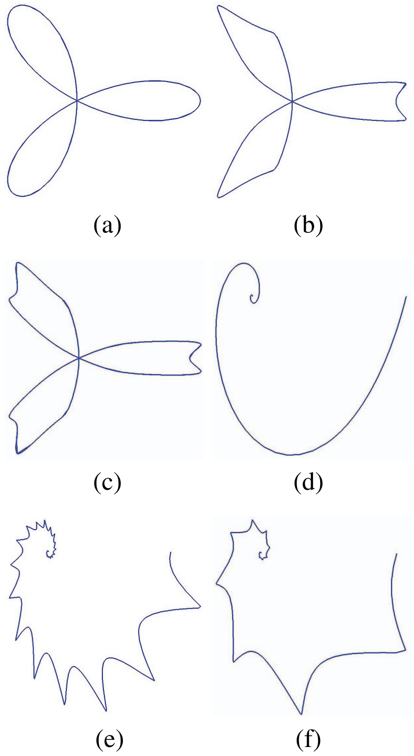

Figure 3. Gielis curves modified by $f(\theta)$. $f(\theta) = \cos(4\theta$, $m = 4$, $n_1 = n_2 = n_3 = 100$, $m = 5$, $n_1 = n_2 = n_3 = 5$ $f(\theta) = \exp(0.5\theta$, $m = 4$, $n_1 = n_2 = n_3 = 100$, $m = 2$, $n_1 = n_2 = n_3 = 100$

IV. Examples

In this section we show some examples. For this purpose we construct some 2D and 3D geometry models by using supershapes given parametrically in the Equations ((8)-(10) respectively.

First, we construct a geometric model of a planar curve $f(\theta)$ by using generalized superformula given parametrically in the Equation (9). For more details the reader is referred to [6]. Figure 3. shows examples of Geilis curves modified by $f(\theta)$.

As a second example, we construct a geometry model of a bean shaped curve and the corresponding surface. The curve corresponding to the geometry of the bean shaped curve is given be by the superformula,

$$r(\theta) = \frac{1}{\left(\left| \frac{\cos(\frac{\theta}{2})}{1} \right|^{11.1909} + \left| \frac{\sin(\frac{\theta}{2})}{2} \right|^{1.3} \right)^{\frac{1}{1.737933}}} \tag{25}$$

The geometry model corresponding the bean shaped surface is given by supershape formula which is described parametrically using the Equation (10). Figure 4. shows the geometry of the bean shaped curve and the corresponding surface.

Finally, in Figure 5 we show examples of some flat and minimal spherical product surfaces discussed in this paper.

V. Conclusion

In this paper, a method of spherical product surface of two 2D curves is investigated. To demonstrate the performance of the proposed method, parameters of superquadrics and supershapes models were constructed from the superellipses and superformulas. Superquadrics and supershapes are solid models that possess simple parametrisations and are capable of representing a wide variety of standard geometric solids as well as smooth shapes in between. This makes them much more convenient for representing rounded, blob-like geometry which are common in nature.

In this paper, using differential geometry we classify

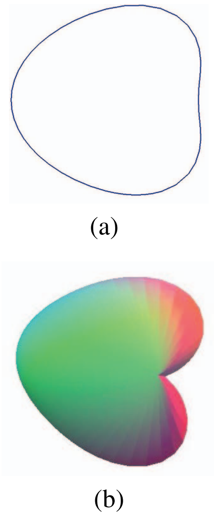

Figure 4. The bean shaped models

Figure 5. Examples of minimal spherical product surfaces

the spherical product surfaces of flat or minimal type. The results we have obtained suggest that we can develop techniques for generating a wide variety geometry which can be defined as mathematical functions. Often this type of geometry generation techniques, where the geometry is defined as a simple mathematical functions, is desirable. For example, function based geometry modelling techniques can represent the geometry of an object with arbitrary level of resolution as opposed to standard mesh models.

As for future work we aim to study the spherical product surfaces on a 3D curve with a 2D curve which will be a surface in E^4. Such a formulation can be utilised for developing techniques for studying time-dependent geometry, for example for the purpose of computer animation.

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References

[1] A. H. Barr, *Global and Local Deformations of Solid Primitives*. Computer Graphics, 18, 1984, 21-30.

[2] T. Bhabhrawala, *Shape Recovery from Medical Image Data Using Extended Superquadrics*. MSc Thesis, State University of New York at Buffalo, December, 2004.

[3] T.O. Binford, *Visual Perception by Computer*. In IEEE Conference on Systems and Controls, 1971.

[4] T. Boult and A. Gross, *Error of Fit Measures for Recovering Parametric Solids*, in International Conferemce pf Computer Vision, 1988.

[5] M. Gardiner, *The Superellipse: A Curve Between the Ellipse and the Rectangle*, Scientific American 213, 1965,222-234.

[6] J. Gielis, *A Generic Geometric Transformation that Unities a Wide Range of Natural and Abstract Shapes.* American Journal of Botany, 90, 2003, 333-338.

[7] J. Gielis, *Methods and Devices for Synthesizing and Analysing Patterns using a Novel Mathematical Operator*. Patent Application USPTO N° 60/133,279, 1999.

[8] J. Gielis, J Beirinchx and Bastianens, *Superquadrics with Rational and Irrational Symmetry*, Symposium on solid modelling and Applications, 2003.

[9] J. Gielis, and T. Gerats, *A Botanical Perspective on Modeling Plants and Plant Shapes,* International Conference on Computing Communications and Control Technologies. Austin TX, Aug. 14-17, 2004.

[10] A. Gray, *Modern Differential Geometry of Curves and Surfaces with Mathematica*, Second Edition, CCR Press, 1997.

[11] A. Gupta and R. Bajcsy, *Surface and volumetric segmentation of range images using biquadrics and superquadrics*. In International Conference on Pattern Recognition, 1, 1992, 158-162.

[12] A. Jaclic, A. Leonardis and F. Solina, *Segmentation and Recovery of Superquadrics*. Kluwer Academic Publishers, Vol. 20, 2000.

[13] G. Hager, *Task-directed computation of qualitative decisions from sensor data*. IEEE Transactions on Robotics and Automations, 10(4), 1994, 415–429.

[14] A. Leonardis, A. Jaklic and F. Solina, *Superquadrics for Segmenting and Modeling Range Data*, IEEE Transactions on Pattern Analysis and Machine Intelligence, 19(11), 1997, 1289-1295.

[15] N. Magnenat-Thalmann, D. Thalmann, *"Image Synthesis: Theory and Practice"*, Springer, 1987.

[16] J.R. Miller, *Geometric Approaches to Nonplanar Quadratic Surface Intersecting Curves*, ACM Transactions on Graphics 6, 1987, 274-307.

[17] A.P. Pentland, *Percettual Organization and the Representation of Natural Form*, Artificial Intelligence, 28, 1986, 293-331.

[18] A. P. Pentland, *Automatic Extraction of Deformable Part Models*. International Journal of Computer Vision, 4, 1990, 107-126.

[19] F. Solina, and R. Bajcsy, *Recovery of Parametric Models from Range Images: The Case for Superquadrics with Global Deformations*, IEEE Transactions on Pattern Analysis and Machine Intelligence, 12(2), 1990, 131-147.

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