Three-Step Iterative Method with Sixth Order Convergence for Solving Nonlinear Equations

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Abstract

The aim of this paper is to construct an efficient iterative method to solve nonlinear equations. This method is obtained from M. Javidi’s method (Appl. Math. Comput. 193 (2007) 360-365), which is third-order. The convergence order of new method is established to six and the efficiency index is 1.5651. The proposed method is compared with the second, third and sixth order methods. Some numerical test problems are given to show the accuracy and fast convergence of the proposed method.

Mathematics Subject Classification: 65N99

Keywords: Nonlinear equations, Newton based methods, Multi-step iterative methods, Convergence analysis, Root finding techniques.

1 Introduction

Solving nonlinear equations is one of the most important and challenging problem in scientific and engineering applications. In this paper, we consider an iterative method to solve nonlinear equation

\[ f(x) = 0 \]  

The well known Newton Raphson’s method is largely used to solve nonlinear equation (1.1) and written as

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0. \]
This is an important and basic method in [6], which converges quadratically.

Recently, a large number of methods which are based on the Newton's method, are proposed to solve nonlinear equations. All these modified methods are in the direction of improving the efficiency index and order of convergence by using lower-order derivatives as possible.

Consider the following iterative method proposed by M. Javidi [3] to construct a new sixth-order method.

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f''(x_n)}{2[f'(x_n)]^3} \]  

This method is third-order and the efficiency index is \(3^{\frac{4}{3}} = 1.4422\).

\section{Derivation of new method}

First we replace \(f''(x_n)\) in (1.3) with a finite difference between first derivatives [4], i.e.

\[ f''(x_n) = \frac{f'(y_n) - f'(x_n)}{y_n - x_n}, \]  

where

\[ y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0. \]  

using 2.1 in 1.3, we obtained an equivalent form:

\[ y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \]  

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \frac{f(x_n)(f'(y_n) - f'(x_n))}{2[f'(x_n)]^2}. \]  

The number of function evaluations in (2.3) and (2.4) is three. To rise the order of convergence we use a Newton’s method step to obtain a new three-step method of order six.

\[ y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \]  

\[ z = x_n - \frac{f(x_n)}{f'(x_n)} + \frac{f(x_n)(f'(y_n) - f'(x_n))}{2[f'(x_n)]^2}, \]  

\[ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \]  

The number of function evaluation in (2.5), (2.6) and (2.7) is also increased by increasing the step. To obtain efficient method it is essential to reduce the number
Three-step iterative method

of function evaluation at each iteration. Now we approximate the first derivative on point $z_n$ with the combination of linear interpolation [7] using two points $(x_n, f'(x_n))$ and $(y_n, f'(y_n))$

$$f'(x) \approx \frac{x - x_n}{y_n - x_n} f'(y_n) + \frac{x - y_n}{x_n - y_n} f'(x_n)$$

(2.8)

to obtain the approximation

$$f'(z_n) \approx -\frac{[f'(x_n)]^2 - 4f'(x_n)f''(y_n) + [f'(y_n)]^2}{2f'(x_n)}$$

(2.9)

This formulation allows us to suggest the following three-step iterative method for solving nonlinear equations.

**Algorithm 1:** For a given $x_0$, calculate the approximation solution $x_{n+1}$, by the iterative scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z = x_n - \frac{f(x_n)}{f'(x_n)} + \frac{f(x_n)(f'(y_n) - f'(x_n))}{2[f'(x_n)]^2},$$

$$x_{n+1} = z_n + \frac{2f(z_n)f'(y_n)}{[f'(x_n)]^2 - 4f'(x_n)f''(y_n) + [f'(y_n)]^2}.$$ (2.10)

### 3 Convergence Analysis

**Theorem:** Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \subseteq R \to R$ for an open interval $I$. If $x_0$ sufficiently close to $\alpha$, then the Algorithm 1 has six order convergence.

**Proof:** Let $e_n = x_n - \alpha$ be the error in the iterate $x_n$. Using Taylor’s series expansion, we get

$$f(x_n) = f'(\alpha) [e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + O(e_n^7)]$$

(3.1)

and

$$f'(x_n) = f'(\alpha) [1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_6 e_n^6 + O(e_n^7)]$$

(3.2)

where $c_k = \frac{f^{(k)}(\alpha)}{k! f'(\alpha)}$ for $k \in N$.

Now

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2) e_n^3 + (4c_4^2 - 7c_2c_3 + 3c_4) e_n^4 + (-8c_2^2 + 20c_2^2 c_3 - 6c_3^2 - 10c_2 c_4 + 4c_5) e_n^5 + (16c_4^5 - 52c_3^3 c_3 + 28c_2^2c_4 - 17c_3c_4 + c_2 (33c_2^2 - 13c_3)) + 5c_6) e_n^6 + O(e_n^7).$$

(3.3)
Initial Approximation

and $6$. The order (efficiency index) of these methods are Algorithm 1(MR) and compare it with the classical Newton Raphson (NR), Wang's method and Homeier's method respectively. The list of test problems are given in Table-1.

<table>
<thead>
<tr>
<th>S. No.</th>
<th>$f(x)$</th>
<th>Initial Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$e^{-x} + \cos x$</td>
<td>1</td>
</tr>
<tr>
<td>2.</td>
<td>$x - 2 - e^{-x}$</td>
<td>3</td>
</tr>
<tr>
<td>3.</td>
<td>$\sqrt{x - x}$</td>
<td>1.5</td>
</tr>
<tr>
<td>4.</td>
<td>$230x^4 + 18x^3 + 9x^2 - 220x - 8$</td>
<td>1.5</td>
</tr>
<tr>
<td>5.</td>
<td>$e^{-x} \sin x + \ln(x^2 + 1) - 3$</td>
<td>3.5</td>
</tr>
</tbody>
</table>

Now we discuss the efficiency index of algorithm 1 by using the definition of [1] as $p^{\frac{q}{2}}$, where $p$ is order of the method and $q$ is the number of function evaluations per iteration required of the method. It is easy to know that the number of function evaluations per iteration required by the methods defined in algorithm 1 is four. So the efficiency index is $6^\frac{4}{2} = 1.5651$, which is better than Newton’s method $2^\frac{5}{2} = 1.4142$, Homeier’s method $3^\frac{5}{2} = 1.4422$ [2], J. Kou’s method $5^\frac{7}{2} = 1.4953$ [4] and X. Wang’s method $6^\frac{5}{2} = 1.4310$ eq. (8) in [8].

### 4 Applications

Now, consider some test problems to illustrate the efficiency of the developed method namely Algorithm 1(MR) and compare it with the classical Newton Raphson (NR), M. Javidi (MJ) [3], S.K Parhi et al (SK)[7], X. Wang (XW) [8] as shown in Table 2. The order (efficiency index) of these methods are 2(1.41412), 3(1.4422), 6(1.5650) and 6(1.4310) respectively. The list of test problems are given in Table-1.
### Table 2

<table>
<thead>
<tr>
<th>Function</th>
<th>Solution</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>NR</td>
</tr>
<tr>
<td>$f_1$</td>
<td>1.746139530408013</td>
<td>4</td>
</tr>
<tr>
<td>$f_2$</td>
<td>2.120028238987641</td>
<td>4</td>
</tr>
<tr>
<td>$f_3$</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>$f_4$</td>
<td>0.95940587420224</td>
<td>7</td>
</tr>
<tr>
<td>$f_5$</td>
<td>1.733207914136137</td>
<td>5</td>
</tr>
</tbody>
</table>

## 5 Conclusions

A sixth order method is proposed to solve nonlinear equations without evaluation of second derivative of the function and it requires two functions and two first derivatives per iteration. Its efficiency index is 1.5651 which is better than the Newton, Homeier [2], J. Kou [5] and X. Wang’s methods. With the help of some test problems, comparison of the obtained results with the existing methods such as the Newton Raphson (NR), M. Javidi (MJ) [3], S.K Parhi(SK) [7], X. Wang (XW) [8] is also given.

## References


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