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Three-Step Iterative Method with Sixth Order Convergence for Solving Nonlinear Equations

Muhammad Rafiullah

Dept. of Mathematics, COMSATS Institute
of Information Technology, Lahore, Pakistan
mrafiullah@ciitlahore.edu.pk

Muhammad Haleem

Dept. of Eletrical Eng., COMSATS Institute
of Information Technology, Lahore, Pakistan
mhaleem@ciitlahore.edu.pk

Abstract

The aim of this paper is to construct an efficient iterative method to solve nonlinear equations. This method is obtained from M. Javidi's method (Appl. Math. Comput. 193 (2007) 360-365), which is third-order. The convergence order of new method is established to six and the efficiency index is 1.5651. The Proposed method is compared with the second, third and sixth order methods. Some numerical test problems are given to show the accuracy and fast convergence of the proposed method.

Mathematics Subject Classification: 65N99

Keywords: Nonlinear equations, Newton based methods, Multi-step iterative methods, Convergence analysis, Root finding techniques.

1 Introduction

Solving nonlinear equations is one of the most important and challenging problem in scientific and engineering applications. In this paper, we consider an iterative method to solve nonlinear equation

$$f(x) = 0 \tag{1.1}$$

The well known Newton Raphson's method is largely used to solve nonlinear equation (1.1) and written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0. \tag{1.2}$$

This is an important and basic method in [6], which converges quadratically.

Recently, a large number of methods which are based on the Newton's method, are proposed to solve nonlinear equations. All these modified methods are in the direction of improving the efficiency index and order of convergence by using lower-order derivatives as possible.

Consider the following iterative method proposed by M. Javidi [3] to construct a new sixth-order method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f''(x_n)}{2[f'(x_n)]^3} \tag{1.3}$$

This method is third-order and the efficiency index is $3^{\frac{1}{3}} = 1.4422$.

2 Derivation of new method

First we replace $f''(x_n)$ in (1.3) with a finite difference between first derivatives [4], i.e.

$$f''(x_n) = \frac{f'(y_n) - f'(x_n)}{y_n - x_n}, \tag{2.1}$$

where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0. \tag{2.2}$$

using 2.1 in 1.3, we obtained an equivalent form:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2.3}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \frac{f(x_n)(f'(y_n) - f'(x_n))}{2[f'(x_n)]^2}. \tag{2.4}$$

The number of function evaluations in (2.3) and (2.4) is three. To rise the order of convergence we use a Newton's method step to obtain a new three-step method of order six.

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2.5}$$

$$z = x_n - \frac{f(x_n)}{f'(x_n)} + \frac{f(x_n)(f'(y_n) - f'(x_n))}{2[f'(x_n)]^2}, \tag{2.6}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \tag{2.7}$$

The number of function evaluation in (2.5), (2.6) and (2.7) is also increased by increasing the step. To obtain efficient method it is essential to reduce the number of function evaluation at each iteration. Now we approximate the first derivative on point z_n with the combination of linear interpolation [7] using two points $(x_n,\ f'(x_n))$ and $(y_n,\ f'(y_n))$

$$f'(x) \approx \frac{x - x_n}{y_n - x_n} f'(y_n) + \frac{x - y_n}{x_n - y_n} f'(x_n) \tag{2.8}$$

to obtain the approximation

$$f'(z_n) \approx -\frac{[f'(x_n)]^2 - 4f'(x_n)f'(y_n) + [f'(y_n)]^2}{2f'(x_n)} \tag{2.9}$$

This formulation allows us to suggest the following three-step iterative method for solving nonlinear equations.

Algorithm 1: For a given x_0, calculate the approximation solution x_{n+1}, by the iterative scheme

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z &= x_n - \frac{f(x_n)}{f'(x_n)} + \frac{f(x_n)\left(f'(y_n) - f'(x_n)\right)}{2\left[f'(x_n)\right]^2}, \\ x_{n+1} &= z_n + \frac{2f(z_n)f'(y_n)}{[f'(x_n)]^2 - 4f'(x_n)f'(y_n) + [f'(y_n)]^2}. \end{aligned} \tag{2.10}$$

3 Convergence Analysis

Theorem: Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \subseteq R \rightarrow R$ for an open interval I. If x_0 sufficiently close to α, then the Algorithm 1 has six order convergence.

Proof: Let $e_n = x_n - \alpha$ be the error in the iterate x_n. Using Taylor's series expansion, we get

$$f(x_n) = f'(\alpha)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)\right] \tag{3.1}$$

and

$$f'(x_n) = f'(\alpha)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_6e_n^6 + O(e_n^7)\right] \tag{3.2}$$

where $c_k = \frac{f^{(k)}(\alpha)}{k!f(\alpha)}$ for $k \in N$.
Now

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ &\quad +(-8c_2^4 + 20c_2^2c_3 - 6c_3^2 - 10c_2c_4 + 4c_5)e_n^5 + (16c_2^5 - 52c_2^3c_3 + \\ &\quad 28c_2^2c_4 - 17c_3c_4 + c_2\left(33c_3^2 - 13c_5\right)) + 5c_6)e_n^6 + O(e_n^7). \end{aligned} \tag{3.3}$$

$$\begin{aligned} f(y_n) &= f'(\alpha)[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ &\quad -2(6c_2^4 - 12c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)e_n^5 + (28c_2^5 - 73c_2^3c_3 + 34c_2^2c_4 \\ &\quad -17c_3c_4 + c_2 37c_3^2 - 13c_5)) + 5c_6)e_n^6 + O(e_n^7)] \end{aligned} \quad (3.4)$$

$$\begin{aligned} f'(y_n) &= f'(\alpha)[1 + 2c_2^2e_n^2 + (4c_2c_3 - 4c_2^3)e_n^3 + c_2(8c_2^3 - 11c_2c_3 + 6c_4)e_n^4 \\ &\quad -4(c_2(4c_2^4 - 7c_2^2c_3 + 5c_2c_4 - 2c_5))e_n^5 + 2(16c_2^6 - 34c_2^4c_3 + 6c_3^3 \\ &\quad +30c_2^3c_4 - 13c_2^2c_5 + c_2 - 8c_3c_4 + 5c_6))e_n^6 + O(e_n^7)] \end{aligned} \quad (3.5)$$

$$\begin{aligned} z &= \frac{1}{2}c_3e_n^3 + (-c_2^3 + \frac{3}{2}c_2c_3 + c_4)e_n^4 + \frac{1}{2}(8c_2^4 + c_2^2c_3 - 6c_3^2 - 4c_2c_4 + 3c_5)e_n^5 \\ &\quad +\frac{1}{2}(-24c_2^5\_17c_2^3c_3 - 4c_2^2c_4 - 17c_3c_4 + 5c_2\left(5c_3^2 - c_5\right) + 4c_6\right) e_n^6 + O(e_n^7) \end{aligned} \quad (3.6)$$

using (3.1) to (3.6) in (2.10), we get

$$e_{n+1} = \frac{1}{4}c_2c_3^2e_n^6 + O(e_n^7)$$

which shows that Algorithm 1 is at least sixth order convergent method.

Now we discuss the efficiency index of algorithm 1 by using the definition of [1] as $p^{\frac{1}{q}}$, where p is order of the method and q is the number of function evaluations per iteration required of the method. It is easy to know that the number of function evaluations per iteration required by the methods defined in algorithm 1 is four. So the efficiency index is $6^{\frac{1}{4}} = 1.5651$, which is better than Newton's method $2^{\frac{1}{2}} = 1.4142$, Homeier's method $3^{\frac{1}{3}} = 1.4422$ [2], J. Kou's method $5^{\frac{1}{4}} = 1.4953$ [4] and X. Wang's method $6^{\frac{1}{5}} = 1.4310$ eq. (8) in [8].

4 Applications

Now, consider some test problems to illustrate the efficiency of the developed method namely Algorithm 1(MR) and compare it with the classical Newton Raphson (NR), M. Javidi (MJ) [3], S.K Parhi et al (SK)[7], X. Wang (XW) [8] as shown in Table 2. The order(efficiency index) of these methods are 2(1.41412), 3(1.4422), 6(1.5650) and 6(1.4310) respectively. The list of test problems are given in Table-1.

Table-1

S. No.	$f(x)$	Initial Approximation
1.	$e^{-x} + \cos x$	1
2.	$x - 2 - e^{-x}$	3
3.	$\sqrt{x} - x$	1.5
4.	$230x^4 + 18x^3 + 9x^2 - 220x - 8$	1.5
5.	$e^{-x} \sin x + \ln(x^2 + 1) - 3$	3.5

Table-2

Function	Solution	Method				
		NR	MJ	SK	XW	MR
$f1$	1.746139530408013	4	3	2	3	2
$f2$	2.120028238987641	4	3	2	3	2
$f3$	1	4	3	2	3	2
$f4$	0.959440587420224	7	5	2	28	2
$f5$	1.733207914136137	5	3	2	3	2

5 Conclusions

A sixth order method is proposed to solve nonlinear equations without evaluation of second derivative of the function and it requires two functions and two first derivatives per iteration. Its efficiency index is 1.5651 which is better than the Newton, Homeier [2], J. Kou [5] and X. Wang's methods. With the help of some test problems, comparison of the obtained results with the existing methods such as the Newton Raphson (NR), M. Javidi (MJ) [3], S.K Parhi(SK) [7], X. Wang (XW) [8] is also given.

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