Pencils of symmetric surfaces in $\mathbb{P}^3_{\mathbb{C}}$

vorgelegt von
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to my parents and my sister

Büschel von symmetrischen Flächen in $\mathbb{P}^3_{\mathbb{C}}$

Zusammenfassung

In der vorliegenden Arbeit betrachte ich einige neue Familien von Flächen in $\mathbb{P}^3_{\mathbb{C}}$ mit vielen gewöhnlichen Doppelpunkten. Sei $\mathbb{C}[x_0, x_1, x_2, x_3]$ der Vektorraum von komplexen Polynomen in vier Veränderlichen. Eine Fläche in $\mathbb{P}^3_{\mathbb{C}}$ ist der Verschwindungsort eines homogenen Polynoms, f, in $\mathbb{C}[x_0, x_1, x_2, x_3]$. Ein Punkt $p \in S$ heißt *singulär* wenn alle partielle Ableitungen der ersten Ordnung verschwinden. Er heißt *Doppelpunkt* wenn es mindestens eine partielle Ableitung der zweiten Ordnung gibt, die nicht in p verschwindet. Schließlich heißt ein singulärer Punkt, p, *gewöhnlicher Doppelpunkt*, wenn man in einer Umgebung von p die Gleichung von S als $x^2 + y^2 + z^2 = 0$ schreiben kann. Sei d der Grad von f, der, nach Definition, der Grad von S ist. Es ist natürlich zu fragen, was die maximale Anzahl, $\mu(d)$, von gewöhnlichen Doppelpunkten ist, die eine Fläche vom Grad d in $\mathbb{P}^3_{\mathbb{C}}$ haben kann, die sonst keine weiteren Singularitäten hat. In der Einleitung zu meiner Doktorarbeit gebe ich die bis jetzt bekannten Ergebnisse für $\mu(d)$ an. Die Idee die hinter vielen Beispielen von Flächen mit vielen gewöhnlichen Doppelpunkten steckt, ist die Folgende: man betrachtet Flächen mit vielen Symmetrien. Man kann diese Idee folgendermaße algebraisch erklären. Man betrachtet eine endliche Gruppe G die auf $\mathbb{C}[x_0, x_1, x_2, x_3]$ operiert. Ein homogenes Polynom, P, vom Grad d heißt *invariant* unter G wenn $g \cdot P = P$, für alle $g \in G$, wobei ich mit $\cdot$ die Gruppenaktion von G bezeichne. Die Nullstellemenge von P in $\mathbb{P}^3_{\mathbb{C}}$ definiert eine symmetrische Fläche. In dieser Weise habe ich einige Büschel von symmetrischen Flächen gefunden. Im ersten Kapitel werden die Symmetrie-Gruppen der platonischen Körper (Tetraeder, Oktaeder, Ikosaeder) beschrieben; eine genauere Beschreibung findet man in [7], [17]. Diese sind Untergruppen T, O, I von SO(3). Mit Hilfe von zwei klassischen Abbildungen aus der Theorie der Lie Gruppen, cf. e.g. [25] S. 77-78, erhält man aus diesen Gruppen Untergruppen von SU(2) und dann von SO(4). Diese bezeichne ich mit G_6, G_8, bzw. G_{12}. Sie operieren auf $\mathbb{C}[x_0, x_1, x_2, x_3]$ und ich untersuche die Vektorräume $\mathbb{C}[x_0, x_1, x_2, x_3]_j^{G_n}$ der homogenen G_n-invarianten Polynome vom Grad j. Im Kapitel 2 werden mit Hilfe der Poincaré-Reihen und eines Theorems von Molien (s. [5], S. 21) die Dimensionen der obigen Räume berechnet. Für $n = 6, 8, 12$ und $j = 6, 8$, bzw. 12 gibt es genau zwei Erzeugende: die vielfache Quadrik $Q_n(x) := (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{\frac{n}{2}}$ und ein anderes Polynom $S_n(x)$. Die expliziten Gleichungen werden am Ende vom Kapitel 2 berechnet. Für die Rechnungen habe ich das Computeralgebra-System MAPLE benutzt. Die zwei invarianten Polynome definieren Büschel von symmetrischen Flächen in $\mathbb{P}^3_{\mathbb{C}}$. Der

Basisort wird im Kapitel 3 berechnet. Es besteht aus $2n$ Geraden, die auf der Quadrik Q_n liegen und Fixgeraden für Elemente in G_n sind. Dabei meine ich punktweise feste Geraden, vgl. Definition 3.1, S. 34. Im Kapitel 4 untersuche ich, dann die singulären Flächen in jedem Büschel. Ihre Anzahl ist genau vier, und jede Fläche hat nur eine G_n-Bahn von gewöhnlichen Doppelpunkten, die auf Fixgeraden für Elemente von G_n liegen. Diese Flächen haben keine weitere Singularitäten. Im Grad 12 findet man eine Fläche mit 600 gewöhnlichen Doppelpunkten. Das bestätigt eine Vermutung von Dr. V. Goryounov in Europroj '96 über die Existenz einer solchen Fläche. Das gibt $600 \leq \mu(12) \leq 645$, wobei die letzte Ungleichung die obere Schranke von Miyaoka für Flächen von Grad 12 ist (s. auch die Tabelle auf Seite iii der Einleitung). Im Kapitel 5 mache ich noch einige Bemerkungen über die totale Anzahl von singulären Punkten in jedem Büschel. Diese hängen mit der Topologie der Büschel zusammen. Schließlich, im Kapitel 6 gebe ich die Tabellen mit den Konfigurationen von Fixgeraden und singulären Punkten, und einige Computer-Bilder der Flächen. Diese sind mit dem Programm SURF von Dr. S. Endraß realisiert worden.

Introduction

In this thesis we deal with surfaces in three dimensional complex projective space $\mathbb{P}^3_{\mathbb{C}}$ with many double points, more precisely, with many nodes. Let $\mathbb{C}[x_0, x_1, x_2, x_3]$ denote the vector space of complex polynomials in four variables. A surface S in $\mathbb{P}^3_{\mathbb{C}}$ is the zero locus of a homogeneous polynomial f in $\mathbb{C}[x_0, x_1, x_2, x_3]$. A point p of S is *singular* if the first derivatives of f vanish at p. It is a *double point* if a derivative of the second order does not vanish at it. Finally, p is a *node* (or *ordinary double point* A_1) if it is singular and locally the equation of f can be put in the form $x^2 + y^2 + z^2 = 0$, i.e. the point "looks like" the vertex of an affine cone. Let d denote the degree of f, which by definition is the degree of S in $\mathbb{P}^3_{\mathbb{C}}$. It is natural to ask what is the maximal number $\mu(d)$ of nodes which can occur on a surface S of degree d. A very easy answer is given in degree $d = 1$, then $\mu(1) = 0$ (plane) and $d = 2$, then $\mu(2) = 1$ (cone). The problem is solved for $3 \leq d \leq 6$ as well. Already at the end of the nineteenth century (1864) it was shown that $\mu(3) = 4$ (Cayley cubic, cf. [23]) and $\mu(4) = 16$ (Kummer surfaces, cf. [18]). Then it was proved that $\mu(5) = 31$ (Togliatti, 1940, cf. [26]; Beauville, 1979, cf. [4]) and $\mu(6) = 65$ (Barth, 1996, cf. [2]; Jaffe & Rubermann, 1996, cf. [15]). The problem is still open in degree $d \geq 7$. In this case there are estimates for the maximal number of nodes. The most recent and the best ones so far are Varchenko's spectral bound (1983, cf. [27]) and Miyaoka's bound (1984, cf. [21]). We recall them, up to degree 12, in the following table

d	7	8	9	10	11	12
$\mu(d) \leq$	104	174	246	360	480	645
	[27]	[21]	[27]	[21]	[27]	[21]

The above given estimates hold for the maximal number $N(d)$ of rational double points that S can have as well. Lower bounds are given by constructing surfaces with as many nodes as possible (resp. as many rational double points as possible). For instance Barth found in 1996 a decic surface with 345 nodes (cf. [2]) and in 1998 Endraß found an octic surface with 168 nodes (cf. [8]), which shows that $\mu(10) \geq 345$ and $\mu(8) \geq 168$. A very useful idea in constructing surfaces with many nodes is to find surfaces with many symmetries. The algebraic idea behind the geometric one is the following: let a group G act on $\mathbb{C}[x_0, x_1, x_2, x_3]$, we say that a homogeneous polynomial p, of degree d, is invariant under the action of G if $g \cdot p = p$ for all $g \in G$, where by $\cdot$ we denote the action of $g \in G$ on p. The equation $\{p = 0\}$ defines a surface of degree d in $\mathbb{P}^3_{\mathbb{C}}$ with the symmetries of G. In this way we found

some new surfaces with many nodes.
This thesis consists of seven chapters. In chapter one we describe the well known symmetry groups of the platonic solids: tetrahedron, octahedron, cube, icosahedron, dodecahedron (cf. e.g. [17]). These are subgroups of SO(3); we denote them by T, O and I (remember that cube and dodecahedron have the same symmetry group as octahedron and icosahedron respectively). Then, using some classical maps from theory of Lie groups (cf. e.g. [25]) we get subgroups of SU(2) and then of SO(4). We denote by G_6, G_8 and G_{12} the subgroups of SO(4) which correspond to T, O and I respectively. We call them bi-polyhedral groups. In chapter two we give the conjugacy classes of elements in these groups and their respective characteristic polynomials, which we use to calculate the *Poincaré series.* More precisely, a matrix $\sigma \in G_n \subseteq$ SO(4) operates on $\mathbb{C}[x_0, x_1, x_2, x_3]$, by $\sigma \cdot f(x_0, x_1, x_2, x_3) := f(\sigma^{-1}(x_0, x_1, x_2, x_3))$. Denote by $\mathbb{C}[x_0, x_1, x_2, x_3]_j^{G_n}$ the vector space of homogeneous polynomials of degree j invariant under the action of G_n. The dimensions of these spaces are the coefficients of the *Poincaré series*

$$p(\mathbb{C}[x_0, x_1, x_2, x_3]^{G_n}, t) := \sum_{j=0}^{\infty} t^j \dim \mathbb{C}[x_0, x_1, x_2, x_3]_j^{G_n}.$$

Using *Molien's theorem* (cf.[5], p. 21), we show in chapter two that in fact we can write

$$p(\mathbb{C}[x_0, x_1, x_2, x_3]^{G_n}, t) = \frac{1}{|G_n|} \sum_{g \in G_n} \frac{1}{\det(\mathbb{I} - g^{-1}t)}.$$

We have invariant polynomials just in even degree, in fact, the Heisenberg group, $\mathcal{H}$, is contained in G_n, $n = 6, 8, 12$, and it is well known that the invariant polynomials under $\mathcal{H}$ are squares in the x_i's (cf. e.g. [14], [16]). As stated at the beginning the zero loci of these polynomials in $\mathbb{P}^3_{\mathbb{C}}$ are surfaces with many symmetries. In particular, in degree $n = 6, 8, 12$ we get pencils of surfaces (i.e. linear system of dimension one) with G_n-symmetries. The generators are the multiple complex quadric $Q_n : (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{\frac{n}{2}} = 0$ and a surface S_n. We calculate the equation of S_n at the end of that chapter. The calculations are complex and require the use of a computer algebra system (MAPLE). These are however difficult in degree $n = 12$. In chapter three we compute the base locus of each pencil. It consists of $2n$ lines, n of each ruling of the complex quadric surface in $\mathbb{P}^3_{\mathbb{C}}$. These are fix lines for the action of some elements of G_n. This means that if $L \subseteq \mathbb{P}^3_{\mathbb{C}}$ is such a line then there is a $\sigma \in G_n$, $\sigma \neq \pm\mathbb{I}$ s.t. $\sigma x = x$ for all $x \in L$. In chapter four we find the singular surfaces in these pencils. The general surface is smooth and the singular ones have just isolated singularities. The latter lie on lines of fix points for the action of some elements of G_n and are not contained in the quadric. The singular points are intersection points of these fix lines, and we show that they are all real points. Then we construct a morphism between the fix lines and $\mathbb{P}^1$ and show that the singular points are its ramification points. In the last part of the chapter, using MAPLE, we find the singular points on the fix lines. We describe, then, how the fix lines meet each other to get the number of singular points on the singular surfaces in the pencils. Finally, we show that they are all nodes. In degree 6 and 8 we have surfaces with at most 48 and 144 nodes respectively. In degree 12 we have a surface with 600 nodes. This improves the previous results of Kreiss, 1955 (cf. [19]) and Chmutov, 1992 (cf. [6]), who found surfaces with 576 double points. The existence of such a surface was already affirmed by Dr. V. Goryunov at Europroj '96, but to our knowledge he never published this result, nor the explicit equation of the surface. We recall in the table below the lower bounds for $N(d)$ so far (up to degree 12):

d	7	8	9	10	11	12
$N(d) \geq$	93	168	216	345	425	600
	[6]	[8]	[6]	[2]	[6]	-

In chapter five we give some remarks on the singular surfaces in the pencils. They in fact do not show anything new but confirm in an interesting way the results of the previous chapters. By a theorem of Atiyah (cf. [1], theorem 1) the Euler-Poincaré characteristic of the blow up of a surface with only nodes and no further singularities is the same of that of a smooth surface of the same degree. Using this fact we find again the total number of nodes in each pencil, which we found already in chapter four. Then, using Morse theory, we describe how the surfaces in the pencils behave close to the singularities. Finally, we give the tables of the fix lines and the singular points on these, and some computer pictures of the singular surfaces in the pencils.
Recently Mukai in [22] announced an application of the groups G_n, considered in this thesis, to moduli spaces $A_{(1,q)}$ of abelian surfaces with $(1, q)$ polarization. In fact, he asserts that the quotient $\mathbb{P}^3/G_n$ is isomorphic to the Satake compactification of $A_{(1,3)}$, $(n = 6)$, resp. of $A_{(1,4)}$ $(n = 8)$. And a certain modification of $\mathbb{P}^3/G_{12}$ is isomorphic to the Satake compactification of $A_{(1,5)}$. The pencils of invariant surfaces descend to pencils on these moduli spaces. It is to be expected, that they admit a description in terms of modular forms.

At this point I would like to thank Prof. W. P. Barth for suggesting me how to find surfaces with many nodes, and for the patience he showed in answering my questions and helping me several times during my work. I also would like to thank Prof. D. van Straten for letting me know about the talk of Dr. V. Goryunov at Europroj '96 and Dr. S. Endraß for helping me with the program SURF to draw the computer pictures.

Contents

1 Rotation Groups

Let SO(n) denote the special orthogonal group consisting of the orthogonal $n \times n$ matrices with determinant one. In this first chapter we describe briefly the rotation groups of the platonic solids (for a more detailed description cf. [7], chapter III; [17], chapter I) in terms of matrices in SO(3). Then we use these groups to get finite subgroups of SO(4) (cf. [25], p. 77-78).

By *rotation group* we mean the group consisting of all the rotations which bring the solid to coincide with itself.
The five platonic solids are tetrahedron, octahedron, cube, icosahedron and dodecahedron, where octahedron and cube, icosahedron and dodecahedron are *reciprocal* (cf. [7], p. 17), so they have the same rotation group. It is useful to start by describing the *Klein four group*, which is contained in each of the previous rotation groups.

1.1 Klein four group

Consider an orthogonal coordinate system x, y, z in $\mathbb{R}^3$. The elements of the rotation group (not the identity) which bring it to coincide with itself are given by rotations of π around x, y, respectively z. In terms of matrices of SO(3) we may write:

$$A_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, A_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, A_3 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These are matrices of period two and we denote by $V := \{\mathbb{I}, A_1, A_2, A_3\}$ the *Klein four group* (cf. [17], §5, p.12).

Let now N_0=number of vertices of a regular polyhedron, N_1=number of its edges, N_2=number of its faces. The order of its rotation group is $2 \cdot N_1$ (cf. [7], p. 47). In what follows we describe the rotation group of the tetrahedron, octahedron and icosahedron.

1.2 Tetrahedral group

A tetrahedron is a regular polyhedron with $N_0 = 4$, $N_1 = 6$ and $N_2 = 4$, so the *tetrahedral group*, T, consists of $2 \cdot N_1 = 2 \cdot 6 = 12$ elements.
Consider the following axes:

-the 4 lines from a vertex through the center of the opposite faces (d-axis);
-the 3 lines from the middle point of an edge through the middle point of the opposite edge (e-axis).
Around each d-axis we have a rotation of period three (i.e. about an angle of $\frac{2}{3}\pi$), therefore we find eight different elements (not the identity) of T. On the other hand around each e-axis we have a rotation of period two (the e-axes all together form an orthogonal system and its rotation group is a Klein four group), so we find $8+3+1=12$ elements .
If we denote the vertices of the tetrahedron by the numbers $1,2,3,4$, we see that each element of T gives an even permutation of these four numbers, and vice versa, therefore we may identify T with A_4, the even permutation group of four objects.
Generators for the group are two elements (not the identity) of the Klein four group and one element of period three.
Now we choose a coordinate system in such a way that the axes x, y, z coincide with the e-axes of the tetrahedron. In this coordinate system the matrices which describe the elements of the Klein four group are the matrices A_1, A_2, A_3 as in section 1.1. Additionally we consider a rotation of period three around a d-axis, which is described by the matrix of SO(3)

$$R_3 := \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then we have $T = \langle A_1, A_2, R_3 \rangle$. In terms of permutations of the tetrahedron's vertices we may identify A_1, A_2, A_3 with (13)(24), (12)(34), (14)(23); R_3 and R_3^2 with (132) and (123).

1.3 Octahedral group

An octahedron is a regular polyhedron with $N_0 = 6$, $N_1 = 12$ and $N_2 = 8$. We call its rotation group *octahedral group* and we denote it by O. It consists of $2 \cdot 12 = 24$ elements.
Consider the following axes:
-the 3 diagonals of the octahedron (d-axis);
-the 6 lines from the middle point of an edge through the middle point of the opposite edge (e-axis);
-the 4 lines from the center of a face through the center of the opposite face (f-axis).
The octahedron admits a rotation of period four around each diagonal (i. e. by an angle of $\frac{\pi}{2}$). In this way we find nine elements of O. Around each

e-axis we have a rotation by an angle of π, and we find six more elements. Finally the rotations around every f-axis give a cyclic subgroup of period three, therefore we have $2 \cdot 4 = 8$ elements. In total we have $9 + 6 + 8 + 1 = 24$ elements.
The centers of the octahedron's faces are the vertices of a cube (the reciprocal polyhedron of the octahedron) and a rotation of the octahedron corresponds to a rotation of the cube into itself, hence to a permutation of the four diagonals. In this way one identifies $O \cong S_4$, the permutation group of four objects. Generators for O are rotations of period two, three and four respectively.
Observe that inside the cube we have two tetrahedra, more precisely a tetrahedron and its *counter-tetrahedron* in a desmic configuration (cf. [14], ch. I, p. 1-2). We denote their vertices respectively by $1, 2, 3, 4$ and $1', 2', 3', 4'$, therefore the diagonals of the cube are $1 := 11'$, $2 := 22'$, $3 := 33'$, and $4 := 44'$. The rotations of the two tetrahedra together give the permutations of the cub's diagonals. In particular the tetrahedral group is contained in the octahedral group. Now we choose a coordinate system such that the axes x, y, z coincide with the three diagonals of the octahedron. We consider two rotations of period two which are described, in this coordinate system, by the matrices A_1, A_2(cf. section 1.1). Enumerating suitably the vertices of the cube and hence the diagonals, the matrices A_1 and A_2 correspond to the permutation (13)(24) and (12)(34) as in section 1.2. Consider now a period three rotation around the cub's diagonal $44'$. The matrix of SO(3), which describes it, is the matrix R_3 of the previous section, and it corresponds to the permutation of the diagonals (132).
Finally we consider a period four rotation around the x-axis, it has the matrix

$$R_4 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

and it corresponds to the permutation (1234) of the diagonals.
Observe that the product

$$R_3 R_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is a matrix of period two and it corresponds to the transposition $(132)(1234) = (34)$. We have $O = \langle A_2, R_3, R_4 \rangle$, because $R_4^2 = A_1$, so we do not need this generator.

1.4 Icosahedral group

Let I denote the rotation group of an icosahedron, the *icosahedral group*. An icosahedron has $N_0 = 12$, $N_1 = 30$ and $N_2 = 20$. Therefore $|I| = 2 \cdot 30 = 60$. Consider the following axes:
-the 6 diagonals of the icosahedron (d-axis);
-the 15 lines from the middle point of an edge through the middle point of the opposite edge (e-axis);
-the 10 lines from the center of a face through the center of the opposite face (f-axis).
Around each d-axis we have a period five rotation (of angle $\frac{2}{5}\pi$), thus we find $4 \cdot 6 = 24$ such rotations.
Around an e-axis and an f-axis we have rotations of period two and three, thus we get $15 + 2 \cdot 10 = 35$ rotations. In total we have $24 + 35 + 1 = 60$ rotations and the icosahedral group is isomorphic to the group A_5, of even permutations of five objects. In fact we have seen that there are fifteen e-axes, and in groups of three they are the e-axis of a tetrahedron (cf. section 1.2). So inside an icosahedron we have five tetrahedra. In particular the tetrahedral group is contained in the icosahedral group. A rotation of the icosahedron gives a permutation of these five tetrahedra. If we consider a rotation of period five, we see that the tetrahedra are cyclically permuted. By a period three rotation two remain invariant and the others are cyclically permuted. Finally by a period two rotation, one remains invariant and the others are switched two by two (for a more detailed description cf. [17], p.19 and [7], p. 49-50).
Generators for I are given by a rotation of period five and two rotations of period two. We choose a coordinate system, in such a way that the x,y,z-axes coincide with three e-axes of the icosahedron, orthogonal to each other. The rotations of period two around the x-axis and y-axis are given respectively by the matrices A_1 and A_2 as before. Enumerating suitably the tetrahedra inside the icosahedron, they correspond to the permutations (13)(24) and (12)(34).
Consider now the rotation of $\frac{2}{5}\pi$ around the d-axis contained in the y, z-plane. We take the icosahedron with vertices in analogous position as [7], p. 52. This axis passes through the vertex $(0, 1, \tau)$, where $\tau = \frac{1}{2}(1+\sqrt{5}) = 2cos(\frac{\pi}{5})$ is the "golden section" number. The matrix R_5 of this rotation, when we rotate in counterclockwise direction, is

$$R_5 := \frac{1}{2}\begin{pmatrix} \tau - 1 & -\tau & 1 \\ \tau & 1 & \tau - 1 \\ -1 & \tau - 1 & \tau \end{pmatrix}.$$

It corresponds to the permutation (13254) of the tetrahedra. We have $I = \langle A_1, A_2, R_5 \rangle$. Moreover, observe that the rotation

$$R_5^2 A_2 = \frac{1}{2} \begin{pmatrix} \tau & -1 & 1-\tau \\ -1 & 1-\tau & -\tau \\ \tau - 1 & \tau & -1 \end{pmatrix}$$

corresponds to the cycle $(12435)(12)(34) = (145)$ and has period three.

1.5 Construction

Let $\mathbb{H}$ denote the real algebra of Hamilton's quaternions. A basis is given by $q_0 = 1,\ q_1 = i,\ q_2 = j,\ q_3 = k$. The multiplication is defined by the following rules $(i, j = 1, 2, 3)$:
(a) $i \in \{0, 1, 2, 3\}$, then $q_i q_0 = q_0 q_i = q_i$
(b) $i \in \{1, 2, 3\}$, then $q_i^2 = -q_0$
(c) If $1 \mapsto i,\ 2 \mapsto j,\ 3 \mapsto k$ is an even permutation of $\{1, 2, 3\}$, then $q_i q_j = -q_j q_i = q_k$.

The quaternions q_0, q_1, q_2, q_3 form an orthonormal basis for the following inner product on $\mathbb{H}$:

$$(\sum_{i=0}^{3} x_i q_i \cdot \sum_{j=0}^{3} y_j q_j) = \sum_{i=0}^{3} x_i y_i, \text{where } x_i, y_i \in \mathbb{R}.$$

The norm $\parallel\ \ \parallel$ defined by this inner product satisfies:

$$\parallel x \cdot y \parallel = \parallel x \parallel \cdot \parallel y \parallel .$$

This implies that the quaternions of length one form a group:

$$\mathbb{H}_1 := \{q \in \mathbb{H};\ \ \parallel q \parallel = 1\}$$

under multiplication. It is a subgroup of the multiplicative group of the non-zero quaternions:

$$\mathbb{H}^* := \{q \in \mathbb{H}; q \neq 0\}.$$

Via the map:

$$\sum_{i=0}^{3} x_i q_i \mapsto \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}$$

the group $\mathbb{H}_1$ is isomorphic to SU(2).

We denote by

$$\rho : \mathrm{SU}(2) \to \mathrm{SO}(3)$$

the epimorphism defined by conjugation. To be precise, let $Q \subset \mathbb{H}$ be the three-dimensional real vector space spanned by q_1, q_2, q_3. Each $q \in \mathbb{H}_1$ via conjugation defines an orthogonal automorphism:

$$\rho_q : \mathbb{H} \to \mathbb{H},\ x \mapsto q \cdot x \cdot q^{-1}$$

Obviously $\rho_q(q_0) = q_0$ and $\rho_q|Q : Q \to Q$ is an automorphism in SO(3). Define:

$$\rho(q) := \rho_q|Q$$

Observe that ρ is a 2:1 morphism, with kernel $\{\mathbb{I}, -\mathbb{I}\}$.
We return, now, to the rotation groups.

1.6 Binary Groups

The pre-image in SU(2) of a rotation group $G \subseteq \mathrm{SO}(3)$ under the map ρ is denoted by $\tilde{G}$. Hence $|\tilde{G}| = 2 \cdot |G|$.

Klein 4-group

First we calculate:

$$\begin{array}{rclrclrcl} \rho(q_1)(q_1) & = & q_1, & \rho(q_1)(q_2) & = & -q_2, & \rho(q_1)(q_3) & = & -q_3, \\ \rho(q_2)(q_1) & = & -q_1, & \rho(q_2)(q_2) & = & q_2, & \rho(q_2)(q_3) & = & -q_3, \\ \rho(q_3)(q_1) & = & -q_1, & \rho(q_3)(q_2) & = & -q_2, & \rho(q_3)(q_3) & = & q_3. \end{array}$$

This shows

$$\rho(q_1) = A_1, \quad \rho(q_2) = A_2, \quad \rho(q_3) = A_3.$$

Denote by $\tilde{V} = \rho^{-1}(V)$, the binary group in SU(2) corresponding to the Klein four group. It consists of the eight quaternions:

$$\{\pm q_i, i = 0, \dots, 3\}$$

Tetrahedral group

Consider the tetrahedral group $T \cong A_4$ and the period six matrix :

$$p_3 := \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix} \in \mathrm{SU}(2),$$

$(p_3^3 = -\mathbb{I})$. We calculate:

$$\rho(p_3)(q_1) = q_3,\ \ \rho(p_3)(q_2) = -q_1,\ \ \rho(p_3)(q_3) = -q_2$$

and this shows that $\rho(p_3) = R_3$. Therefore the binary group $\tilde{A}_4 = \rho^{-1}(T)$ is spanned by q_1, q_2, p_3 and $|\tilde{A}_4| = 24$.

Octahedral group

For the octahedral group $O \cong S_4$, we consider the period eight matrix :

$$p_4 := \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \in \mathrm{SU}(2)$$

$(p_4^4 = -\mathbb{I})$, and

$$\rho(p_4)(q_1) = q_1,\ \ \rho(p_4)(q_2) = q_3,\ \ \rho(p_4)(q_3) = -q_2,$$

therefore $\rho(p_4) = R_4$. Let $\tilde{S}_4 \subseteq \mathrm{SU}(2)$ be the binary group which corresponds to O, then it is spanned by q_1, q_2, p_3, p_4 and $|\tilde{S}_4| = 48$.

Icosahedral group

Finally consider the icosahedral group $I \cong A_5$ and the matrix:

$$p_5 := \frac{1}{2}\begin{pmatrix} \tau & \tau-1+i \\ 1-\tau+i & \tau \end{pmatrix} \in \mathrm{SU}(2)$$

(remember $\tau = \frac{1}{2}(1+\sqrt{5}) = 2cos(\frac{\pi}{5})$). One computes $p_5^5 = -\mathbb{I}$, so p_5 has order 10. It is easy to check that $\rho(p_5) = R_5$.
Let $\tilde{A}_5 = \rho^{-1}(I)$, then $\tilde{A}_5 = \langle q_1, q_2, p_5\rangle$ and $|\tilde{A}_5| = 120$.

1.7 The bi-polyhedral groups in SO(4)

For $q, q' \in \mathbb{H}_1$ the automorphism:

$$\sigma(q,q') : \mathbb{H} \to \mathbb{H}, x \mapsto q \cdot x \cdot (q')^{-1}$$

is orthogonal. This defines an epimorphism (cf. [25], p. 78)

$$\sigma : \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4) \tag{1}$$

with kernel

$$\{(\mathbb{I}, \mathbb{I}), (-\mathbb{I}, -\mathbb{I})\}$$

Whenever $G_1, G_2 \subseteq \mathrm{SU}(2)$ are finite subgroups, then $\sigma(G_1, G_2)$ is a finite subgroup of $\mathrm{SO}(4)$, of order $\frac{1}{2}|G_1| \cdot |G_2|$. In this section we want to determine generators for the groups $\sigma(G_1, G_2)$, when $G_1 = G_2$ equals $\tilde{V}$, $\tilde{A}_4$, $\tilde{S}_4$ or $\tilde{A}_5$.

Remark 1.1 Observe that if $q \in \tilde{G} \subseteq \mathrm{SU}(2)$ ($\tilde{G} = \tilde{V}$, $\tilde{A}_4$, $\tilde{S}_4$ or $\tilde{A}_5$) and $q^m = \pm\mathbb{I}$ then the same holds for $\sigma(q, \mathbb{I})$ resp. $\sigma(\mathbb{I}, q)$. In fact $\sigma(q, \mathbb{I})^m = \sigma(q^m, \mathbb{I}) = \sigma(\pm\mathbb{I}, \mathbb{I}) = \pm\mathbb{I}$.

$G_1 = G_2 = \tilde{V}$.

We compute the images under σ of the generators

$$(q_1, \mathbb{I}), \quad (q_2, \mathbb{I}), \quad (\mathbb{I}, q_1), \quad (\mathbb{I}, q_2)$$

of $\tilde{V} \times \tilde{V}$. By definition $\sigma(q_i, \mathbb{I}) : x \mapsto q_i \cdot x$ and $\sigma(\mathbb{I}, q_i) : x \mapsto x \cdot q_i^{-1}$. On the elements of the basis:

$$\begin{array}{lcrlcrlcr}
\sigma(q_1,\mathbb{I})(q_0) &=& q_1, & \sigma(q_1,\mathbb{I})(q_1) &=& -q_0, & \sigma(q_1,\mathbb{I})(q_2) &=& q_3, \\
\sigma(q_1,\mathbb{I})(q_3) &=& -q_2, & \sigma(q_2,\mathbb{I})(q_0) &=& q_2, & \sigma(q_2,\mathbb{I})(q_1) &=& -q_3, \\
\sigma(q_2,\mathbb{I})(q_2) &=& -q_0, & \sigma(q_2,\mathbb{I})(q_3) &=& q_1, & \sigma(\mathbb{I},q_1)(q_0) &=& -q_1, \\
\sigma(\mathbb{I},q_1)(q_1) &=& q_0, & \sigma(\mathbb{I},q_1)(q_2) &=& q_3, & \sigma(\mathbb{I},q_1)(q_3) &=& -q_2, \\
\sigma(\mathbb{I},q_2)(q_0) &=& -q_2, & \sigma(\mathbb{I},q_2)(q_1) &=& -q_3, & \sigma(\mathbb{I},q_2)(q_2) &=& q_0, \\
\sigma(\mathbb{I},q_2)(q_3) &=& q_1. & & & & & &
\end{array}$$

We call the group $\sigma(\tilde{V} \times \tilde{V})$ *Heisenberg group*, and we denote it by $\mathcal{H}$. From these calculations and remark 1.1, in the basis q_0, q_1, q_2, q_3 generators for $\mathcal{H}$ are given by the following period four matrices:

$$\sigma_1 := \sigma(q_1, \mathbb{I}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sigma_2 := \sigma(q_2, \mathbb{I}) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\sigma_3 := \sigma(\mathbb{I}, q_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sigma_4 := \sigma(\mathbb{I}, q_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We have $|\mathcal{H}| = 32$.

$G_1 = G_2 = \tilde{A}_4$.

Since $\tilde{A}_4 = \langle q_1, q_2, p_3 \rangle$, it remains to calculate the image of $\sigma(p_3, \mathbb{I})$ and $\sigma(\mathbb{I}, p_3)$. With calculations as before we find the following period six matrices of SO(4):

$$\pi_3 := \sigma(p_3, \mathbb{I}) = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\pi_3' := \sigma(\mathbb{I}, p_3) = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix}.$$

Let G_6 denote $\sigma(\tilde{A}_4 \times \tilde{A}_4)$. Generators of G_6 are the matrices σ_1,σ_2,σ_3,σ_4,π_3,π_3' and $|G_6| = 288$.

$G_1 = G_2 = \tilde{S}_4$.

Let now $G_8 := \sigma(\tilde{S}_4 \times \tilde{S}_4)$. Generators of G_8 are the matrices σ_2,σ_4,π_3,π_3' and the matrices $\sigma(p_4, \mathbb{I})$,$\sigma(\mathbb{I}, p_4)$. Here the last two matrices are the following period eight matrices:

$$\pi_4 := \sigma(p_4, \mathbb{I}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\pi_4' := \sigma(\mathbb{I}, p_4) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

and $|G_8| = 1152$.

$G_1 = G_2 = \tilde{A}_5$.

Consider now the bi-polyhedral icosahedral group $G_{12} := \sigma(\tilde{A}_5 \times \tilde{A}_5)$. Generators of G_{12} are the matrices σ_1,σ_2,σ_3,σ_4,$\sigma(p_5, \mathbb{I})$ and $\sigma(\mathbb{I}, p_5)$. We already know the first four matrices, the others are the following period ten matrices:

$$\pi_5 \;:=\; \sigma(p_5, \mathbb{I}) \;=\; \frac{1}{2}\begin{pmatrix} \tau & 0 & 1-\tau & -1 \\ 0 & \tau & -1 & \tau-1 \\ \tau-1 & 1 & \tau & 0 \\ 1 & 1-\tau & 0 & \tau \end{pmatrix},$$

$$\pi_5' \;:=\; \sigma(\mathbb{I}, p_5) \;=\; \frac{1}{2}\begin{pmatrix} \tau & 0 & \tau-1 & 1 \\ 0 & \tau & -1 & \tau-1 \\ 1-\tau & 1 & \tau & 0 \\ -1 & 1-\tau & 0 & \tau \end{pmatrix}.$$

with $\tau := \frac{1}{2}(1+\sqrt{5}) = 2cos(\frac{\pi}{5})$ and $|G_{12}| = 7200$.

The notations G_6, G_8, G_{12} for the bi-polyhedral groups are maybe not standard, but they will be convenient later.
In the following tables we resume the results of this chapter. We give the groups in SO(3), SU(2), SO(4) with their order and the generators with the respective periods:

Groups	Order	Generators	Periods
V	4	A_1, A_2	2, 2
$T \cong A_4$	12	A_1, A_2, R_3	2, 2, 3
$O \cong S_4$	24	A_2, R_3, R_4	2, 3, 4
$I \cong A_5$	60	A_1, A_2, R_5	2, 2, 5

Groups	Order	Generators	Periods
$\tilde{V}$	8	q_1, q_2	4, 4
$\tilde{A}_4$	24	q_1, q_2, p_3	4, 4, 6
$\tilde{S}_4$	48	q_2, p_3, p_4	4, 6, 8
$\tilde{A}_5$	120	q_1, q_2, p_5	4, 4, 10

Groups	Order	Generators	Periods
$\mathcal{H}$	32	σ_1, σ_2, σ_3, σ_4	4, 4, 4, 4
G_6	288	σ_1, σ_2, σ_3, σ_4, π_3, π_3'	4, 4, 4, 4, 6, 6
G_8	1152	σ_2, σ_4, π_3, π_3', π_4, π_4'	4, 4, 6, 6, 8, 8
G_{12}	7200	σ_1, σ_2, σ_3, σ_4, π_5, π_5'	4, 4, 4, 4, 10, 10

2 Invariant polynomials

The subgroups of SO(4) given in chapter 1 act on $\mathbb{C}[x_0, x_1, x_2, x_3]$. The aim of this chapter is to give the dimension of the vector spaces of homogeneous polynomials invariant under the action of G_6, G_8, G_{12} and to calculate the invariant polynomials in degree 6, 8 and 12 respectively.

2.1 Conjugacy Classes

In this section we give the conjugacy classes of the elements in $G_n \subseteq \mathrm{SO}(4)$ under G_n, $n = 6, 8, 12$, and under $\mathrm{GL}(4, \mathbb{C})$.
Let G denote A_4, S_4 or A_5. We have seen that $T \cong A_4$, $O \cong S_4$, $I \cong A_5$. In other words we have representations:

$$\varphi : G \longrightarrow \mathrm{SO}(3, \mathbb{R})$$

and these are group homomorphisms. Hence if $g, g' \in G$ are conjugate in G, this means that there is an $h \in G$ with $g' = hgh^{-1}$. Then $\varphi(g') = \varphi(h)\varphi(g)\varphi(h)^{-1}$ and $\varphi(g')$, $\varphi(g)$ are conjugate resp. in T, O, $I \subseteq \mathrm{SO}(3)$ too. Keeping in mind the table of the conjugacy classes of A_4, S_4 and A_5 (cf. e.g. [9]), in the following table we give representatives of the conjugacy classes of the groups V, T, O, I under the groups themselves and the respective sizes (number of elements in the classes):

Groups	Conjugacy Classes	Size
V	$\mathbb{I}$, A_2	1,3
$T \cong A_4$	$\mathbb{I}$, A_2, R_3, R_3^2	1, 3, 4, 4
$O \cong S_4$	$\mathbb{I}$, A_2, R_3, R_4, R_3R_4	1, 3, 8, 6, 6
$I \cong A_5$	$\mathbb{I}$, A_2, R_5, R_5^2, $R_5^2A_2$	1, 15, 12, 12, 20

In $\mathrm{GL}(3, \mathbb{C})$ the matrices R_3, R_3^2 are conjugate. In fact, they have the same eigenvalues 1, ω, ω^2, with $\omega := e^{\frac{2\pi i}{3}}$. Analogously, the matrices A_2 and R_3R_4 are conjugate in $\mathrm{GL}(3, \mathbb{C})$.
On the contrary, although the matrices R_5 and R_5^2 have the same order, they are not conjugate in $\mathrm{GL}(3, \mathbb{C})$. In fact, they have different characteristic polynomials:

$$\begin{aligned}
\det(R_5 - t\mathbb{I}) &= \tfrac{1}{2}(2t^2 - t\sqrt{5} + t + 2)(t - 1), \\
\det(R_5^2 - t\mathbb{I}) &= \tfrac{1}{2}(2t^2 + t\sqrt{5} + t + 2)(t - 1).
\end{aligned}$$

In the following table we give the conjugacy classes under $\mathrm{GL}(3,\mathbb{C})$:

Group	Conjugacy Classes	Size
V	$\mathbb{I}$, A_2	1,3
$T \cong A_4$	$\mathbb{I}$, A_2, R_3	1, 3, 8
$O \cong S_4$	$\mathbb{I}$, A_2, R_3, R_4	1, 9, 8, 6
$I \cong A_5$	$\mathbb{I}$, A_2, R_5, R_5^2, $R_5^2 A_2$	1, 15, 12, 12, 20

Proposition 2.1 *Let $G \subseteq \mathrm{SO}(3)$ denote one of the groups V, T, O, I, and let $g \in G$. To each conjugacy class $[g]$ under G, we find two conjugacy classes $[+\tilde{g}]$, $[-\tilde{g}]$ in $\tilde{G} = \rho^{-1}(G)$, unless $g \in V$ or $g \in O$ and using the identification of O with S_4, g corresponds to an odd permutation of order 2. In these cases $[+\tilde{g}]=[-\tilde{g}]$.*

Proof. If $\tilde{g}$ and $-\tilde{g} \in \tilde{G}$ are conjugate then they have the same trace: $\mathrm{tr}(\tilde{g}) = \mathrm{tr}(-\tilde{g}) = -\mathrm{tr}(\tilde{g})$. From this follows $\mathrm{tr}(\tilde{g}) = 0$. This is the case only when $\tilde{g} \in \tilde{V}$ or $\tilde{g} \in \tilde{S}_4$ and $\tilde{g} \in [p_3 p_4]$. Otherwise $\mathrm{tr}(\tilde{g}) \neq 0$. We are left to prove that $-q_2 \in [q_2]$ and $-p_3 p_4 \in [p_3 p_4]$. We have

$$p_3 p_4 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} i & i \\ i & -i \end{pmatrix} = \tfrac{1}{\sqrt{2}}(q_1 + q_3).$$

By the multiplication rules $q_i^{-1} = -q_i$, $i = 0,1,2,3$, so we get $q_1 q_2 q_1^{-1} = q_1 q_2 (-q_1) = -q_3 q_1 = -q_2$ and $q_2 \frac{1}{\sqrt{2}}(q_1 + q_3)(-q_2) = -\frac{1}{\sqrt{2}}(q_1 + q_3)$. □

We specify the matrices

$$p_3^2 = \tfrac{1}{2} \begin{pmatrix} -1+i & -1+i \\ 1+i & -1-i \end{pmatrix}, \qquad p_5^2 = \tfrac{1}{2} \begin{pmatrix} \tau - 1 & 1 + i\tau \\ -1 + i\tau & \tau - 1 \end{pmatrix},$$

$$p_5^2 q_2 = \tfrac{1}{2} \begin{pmatrix} -1 - i\tau & \tau - 1 \\ -\tau + 1 & -1 + i\tau \end{pmatrix}.$$

In the following table we give the groups, the conjugacy classes under $\tilde{G}$ and the respective size. We write $\pm q$ to indicate the representatives $+q$, $-q$. Hence we write n, n for the sizes of the respective conjugacy classes.

Groups	Conjugacy Classes	Size
$\tilde{V}$	$\pm\mathbb{I}$, q_2	1,1; 6
$\tilde{A}_4$	$\pm\mathbb{I}$, q_2, $\pm p_3$, $\pm p_3^2$	1,1; 6; 4,4; 4,4
$\tilde{S}_4$	$\pm\mathbb{I}$, q_2, $\pm p_3$, $\pm p_4$, $p_3 p_4$	1,1; 6; 8,8; 6,6; 12
$\tilde{A}_5$	$\pm\mathbb{I}$, q_2, $\pm p_5$, $\pm p_5^2$, $\pm p_5^2 q_2$	1,1; 30; 12,12; 12,12; 20,20

In the next table, we give the eigenvalues of these matrices. We use the following abbreviations for certain roots of unity: $\omega := e^{\frac{2\pi i}{3}}$, $\varepsilon := e^{\frac{2\pi i}{5}}$, $\gamma := e^{\frac{2\pi i}{8}}$.

Matrix	Order	Eigenvalues
q_2	4 with $q_2^2 = -\mathbb{I}$	i, $-i$
p_3	6 with $p_3^3 = -\mathbb{I}$	$-\omega$,$-\omega^2$
p_3^2	3	ω,ω^2
p_4	8 with $p_4^4 = -\mathbb{I}$	γ, γ^7
$p_3 p_4$	4 with $(p_3 p_4)^2 = -\mathbb{I}$	i, $-i$
p_5	10 with $p_5^5 = -\mathbb{I}$	$-\varepsilon^2$, $-\varepsilon^3$
p_5^2	5	ε, ε^4
$p_5^2 q_2$	3	ω, ω^2

Lemma 2.1 *The matrices p_3 and $-p_3^2$, (resp. $-p_3$ and p_3^2) are conjugate in* SU(2)*, the matrices q_2 and $p_3 p_4$ are there conjugate too.*

Proof. The proof follows by the table above, in fact these matrices have two by two the same eigenvalues. □

The conjugacy classes of $\tilde{G}$ under SU(2) are:

Groups	Conjugacy Classes	Size
$\tilde{V}$	$\pm\mathbb{I}$, q_2	1,1; 6
$\tilde{A}_4$	$\pm\mathbb{I}$, q_2, $\pm p_3$	1,1; 6; 8,8
$\tilde{S}_4$	$\pm\mathbb{I}$, q_2, $\pm p_3$, $\pm p_4$	1,1; 18; 8,8; 6,6
$\tilde{A}_5$	$\pm\mathbb{I}$, q_2, $\pm p_5$, $\pm p_5^2$, $\pm p_5^2 q_2$	1,1; 30; 12,12; 12,12; 20,20

The tensor product of $\mathbb{C}^2$ with itself can be identified with the space of complex 2×2-matrices. The tensors $v \otimes w \in \mathbb{C}^2 \otimes \mathbb{C}^2$ can be identified with the zero determinant matrices, via

$$\begin{array}{lcl} \mathbb{C}^2 \times \mathbb{C}^2 & \longrightarrow & \mathbb{C}^4 \\ ((x_1, x_2), (y_1, y_2)) & \longmapsto & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{pmatrix} \end{array} \tag{2}$$

Let $\sigma(p,q) \in G_n$, $n = 6, 8, 12$, and $v_1 \otimes v_2 \in \mathbb{C}^2 \otimes \mathbb{C}^2$. Then we have

$$\begin{array}{rcl} \sigma(p,q)(v \otimes w) & = & p(v \otimes w) q^{-1} \\ & = & p(v \cdot w^t)\bar{q}^t \\ & = & pv \otimes \bar{q}w. \end{array}$$

Using these facts we give

Lemma 2.2 *The eigenvalues, resp. the eigenvectors, of $\sigma(p,q)$ are the products, resp. the tensor products, of the eigenvalues, resp. of the eigenvectors, of p and $\bar{q}$.*

Proof. Let α, $\bar{\alpha}$, v_1, v_2 and β, $\bar{\beta}$, w_1, w_2 the eigenvalues and the eigenvectors of p, resp. q. We have

$$\begin{array}{rcl} \sigma(p,q)(v_1 \otimes \bar{w}_1) & = & pv_1 \otimes \bar{q}\bar{w}_1 \\ & = & \alpha\bar{\beta}(v_1 \otimes \bar{w}_1). \end{array}$$

So $v_1 \otimes \bar{w}_1$ is eigenvector of $\sigma(p,q)$ for the eigenvalue $\alpha\bar{\beta}$. Analogously $v_1 \otimes \bar{w}_2$, $v_2 \otimes \bar{w}_1$, $v_2 \otimes \bar{w}_2$ have eigenvalues respectively $\alpha\beta$, $\bar{\alpha}\bar{\beta}$, $\bar{\alpha}\beta$.

□

We consider now the conjugacy classes of the bi-polyhedral groups $\sigma(\tilde{G} \times \tilde{G}) \subseteq \mathrm{SO}(4)$.

In SU(2)× SU(2) we have the direct product subgroup $\tilde{G} \times \tilde{G}$ and recall that the conjugacy classes here are the products of the conjugacy classes in $\tilde{G}$:

$$[(g_1, g_2)] \quad = \quad [g_1] \times [g_2].$$

In particular the size of the conjugacy class $|[(g_1, g_2)]|$ is the product $|[g_1]| \cdot |[g_2]|$.

Lemma 2.3 *The conjugacy classes in G_n, under G_n, $n = 6, 8, 12$ are the images via the map σ of the conjugacy classes in $\tilde{A}_4 \times \tilde{A}_4$, $\tilde{S}_4 \times \tilde{S}_4$ and $\tilde{A}_5 \times \tilde{A}_5$.*

Lemma 2.4 *If $p, q \in \tilde{G} \subseteq \mathrm{SU}(2)$, then $\sigma(p, q)$ and $\sigma(q, p)$ belong to the same conjugacy class under* $\mathrm{GL}(4, \mathbb{R})$.

Proof. The automorphism

$$\mathbb{H} \to \mathbb{H}, q \mapsto \bar{q}^t$$

has matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \mathrm{GL}(4, \mathbb{R}).$$

In fact, for an $x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{H} \cong \mathbb{R}^4$, one has

$$Cx = \begin{pmatrix} x_0 \\ -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} = \begin{pmatrix} x_0 - ix_1 & -x_2 - ix_3 \\ x_2 + ix_3 & x_0 + ix_1 \end{pmatrix} = \bar{x}^t.$$

Moreover, observe that $C^{-1} = C$. Now

$$C\sigma(q, p)Cx \quad = \quad C(q\bar{x}^t\bar{p}^t) \quad = \quad (\overline{q\bar{x}^t\bar{p}^t})^t \quad = \quad px\bar{q}^t \quad = \quad pxq^{-1} \quad = \quad \sigma(p, q)x.$$

This shows that $\sigma(p, q)$ and $\sigma(q, p)$ are in the same conjugacy class under $\mathrm{GL}(4, \mathbb{R})$, so in particular under $\mathrm{GL}(4, \mathbb{C})$. □

Proposition 2.2 *Let* $c_i \in \tilde{G}$. *The matrices* $\sigma(p_3, c_i)$ *and* $\sigma(-p_3^2, c_i)$ *(resp.* $\sigma(-p_3, c_i)$ *and* $\sigma(p_3^2, c_i)$*),* $\sigma(q_2, c_i)$ *and* $\sigma(p_3 p_4, c_i)$ *belong to the same conjugacy class under* $\mathrm{GL}(4, \mathbb{C})$.

Proof. The matrices p_3 and $-p_3^2$ have the same eigenvalues, so $\sigma(p_3, c_i)$ and $\sigma(-p_3^2, c_i)$ have the same eigenvalues too (cf. lemma 2.2). Moreover they are diagonalizable. Therefore they belong to the same conjugacy class in $\mathrm{GL}(4, \mathbb{C})$. A similar proof works for the other matrices. □

The number of elements in G_n conjugate to a fix element $\sigma(g_1, g_2)$ depends on the group action. In fact $\sigma(p, q) = C\sigma(q, p)C^{-1}$ shows that $\sigma(p, q)$ and $\sigma(q, p)$ are always conjugate under $\mathrm{GL}(4, \mathbb{C})$, but for $[p] \neq [q]$ are not conjugate under G_n. The precise number of elements conjugate under G_n resp. under $\mathrm{GL}(4, \mathbb{C})$ are given in the following table. In the first column the conjugacy classes are given under $\tilde{G}$.

g_1, g_2	$\tilde{G}$	$\mathrm{GL}(4, \mathbb{C})$
$[g_1] \neq [g_2]$ $[g_1],[g_2] \neq [q_2],[p_3 p_4]$	$\lvert[g_1]\rvert \cdot \lvert[g_2]\rvert$	$2\lvert[g_1]\rvert \cdot \lvert[g_2]\rvert$
$[g_1] = [g_2] \neq [q_2],[p_3 p_4]$	$\lvert[g_1]\rvert^2$	$\lvert[g_1]\rvert^2$
$[g_1] \neq [g_2]$ $[g_1],[g_2] = [q_2],[p_3 p_4]$	$\frac{1}{2}\lvert[g_1]\rvert \cdot \lvert[g_2]\rvert$	$\lvert[g_1]\rvert \cdot \lvert[g_2]\rvert$
$[g_1] = [g_2] = [q_2]$ or $[p_3 p_4]$	$\frac{1}{2}\lvert[g_1]\rvert^2$	$\frac{1}{2}\lvert[g_1]\rvert^2$

We are now ready to give the tables of the conjugacy classes. In **1)**, **2)**, **3)** we give the conjugacy classes of the elements of the groups G_n under the groups themselves with their respective size.

1) Bi-tetrahedral group

$\sigma(-,-)$	$\mathbb{I}$	$-\mathbb{I}$	q_2	p_3	$-p_3$	p_3^2	$-p_3^2$
$\mathbb{I}$	$\sigma(\mathbb{I},\mathbb{I})$ 1	$-\sigma(\mathbb{I},\mathbb{I})$ 1	σ_4 6	π_3' 4	$-\pi_3'$ 4	$\pi_3'^2$ 4	$-\pi_3'^2$ 4
q_2	σ_2 6		σ_{24} 18	$\sigma_2\pi_3'$ 24		$\sigma_2\pi_3'^2$ 24	
p_3	π_3 4	$-\pi_3$ 4	$\pi_3\sigma_4$ 24	$\pi_3\pi_3'$ 16	$-\pi_3\pi_3'$ 16	$\pi_3\pi_3'^2$ 16	$-\pi_3\pi_3'^2$ 16
p_3^2	π_3^2 4	$-\pi_3^2$ 4	$\pi_3^2\sigma_4$ 24	$\pi_3^2\pi_3'$ 16	$-\pi_3^2\pi_3'$ 16	$\pi_3^2\pi_3'^2$ 16	$-\pi_3^2\pi_3'^2$ 16

2) Bi-octahedral group

$\sigma(-,-)$	$\mathbb{I}$	$-\mathbb{I}$	q_2	p_3	$-p_3$
$\mathbb{I}$	$\sigma(\mathbb{I},\mathbb{I})$ 1	$-\sigma(\mathbb{I},\mathbb{I})$ 1	σ_4 6	π_3' 8	$-\pi_3'$ 8
q_2	σ_2 6		σ_{24} 18	$\sigma_2\pi_3'$ 48	
p_3	π_3 8	$-\pi_3$ 8	$\pi_3\sigma_4$ 48	$\pi_3\pi_3'$ 64	$-\pi_3\pi_3'$ 64
p_4	π_4 6	$-\pi_4$ 6	$\pi_4\sigma_4$ 36	$\pi_4\pi_3'$ 48	$-\pi_4\pi_3'$ 48
p_3p_4	$\pi_3\pi_4$ 12		$\pi_3\pi_4\sigma_4$ 36	$\pi_3\pi_4\pi_3'$ 96	

$\sigma(-,-)$	p_4	$-p_4$	p_3p_4
$\mathbb{I}$	π_4' 6	$-\pi_4'$ 6	$\pi_3'^2\pi_4'$ 12
q_2	$\sigma_2\pi_4'$ 36		$\sigma_2\pi_3'\pi_4'$ 36
p_3	$\pi_3\pi_4'$ 48	$-\pi_3\pi_4'$ 48	$\pi_3\pi_3'\pi_4'$ 96
p_4	$\pi_4\pi_4'$ 36	$-\pi_4\pi_4'$ 36	$\pi_4\pi_3'\pi_4'$ 72
p_3p_4	$\pi_3\pi_4\pi_4'$ 36	$-\pi_3\pi_4\pi_4'$ 36	$\pi_3\pi_4\pi_3'\pi_4'$ 72

3) Bi-icosahedral group

$\sigma(-,-)$	$\mathbb{I}$	$-\mathbb{I}$	q_2	$p_5^2 q_2$	$-p_5^2 q_2$
$\mathbb{I}$	$\sigma(\mathbb{I},\mathbb{I})$ 1	$-\sigma(\mathbb{I},\mathbb{I})$ 1	σ_4 30	$\pi_5'^2\sigma_4$ 20	$-\pi_5'^2\sigma_4$ 20
q_2	σ_2 30		σ_{24} 450	$\sigma_2\pi_5'\sigma_4$ 600	
$p_5^2 q_2$	$\pi_5^2\sigma_2$ 20	$-\pi_5^2\sigma_2$ 20	$\pi_5^2\sigma_2\sigma_4$ 600	$\pi_5^2\sigma_2\pi_5'^2\sigma_4$ 400	$-\pi_5^2\sigma_2\pi_5'^2\sigma_4$ 400
p_5	π_5 12	$-\pi_5$ 12	$\pi_5\sigma_4$ 360	$\pi_5\pi_5'^2\sigma_4$ 240	$-\pi_5\pi_5'^2\sigma_4$ 240
p_5^2	π_5^2 12	$-\pi_5^2$ 12	$\pi_5^2\sigma_4$ 360	$\pi_5^2\pi_5'^2\sigma_4$ 240	$-\pi_5^2\pi_5'^2\sigma_4$ 240

$\sigma(-,-)$	p_5	$-p_5$	p_5^2	$-p_5^2$
$\mathbb{I}$	π_5' 12	$-\pi_5'$ 12	$\pi_5'^2$ 12	$-\pi_5'^2$ 12
q_2	$\sigma_2\pi_5'$ 360		$\sigma_2\pi_5'^2$ 360	
$p_5^2 q_2$	$\pi_5^2\sigma_2\pi_5'$ 240	$-\pi_5^2\sigma_2\pi_5'$ 240	$\pi_5^2\sigma_2\pi_5'^2$ 240	$-\pi_5^2\sigma_2\pi_5'^2$ 240
p_5	$\pi_5\pi_5'$ 144	$-\pi_5\pi_5'$ 144	$\pi_5\pi_5'^2$ 144	$-\pi_5\pi_5'^2$ 144
p_5^2	$\pi_5^2\pi_5'$ 144	$-\pi_5^2\pi_5'$ 144	$\pi_5^2\pi_5'^2$ 144	$-\pi_5^2\pi_5'^2$ 144

In the last four tables we give the conjugacy classes under $\mathrm{GL}(4,\mathbb{C})$, the respective size and the characteristic polynomials.

$\mathcal{H}$	Conjugacy Classes	Size	Characteristic Polynomials
	σ_2	12	$(t^2+1)^2$
	σ_{24}	18	$(t^2-1)^2$
	$\pm\mathbb{I}$	1,1	$(t\mp 1)^4$

G_6	Conjugacy Classes	Size	Characteristic Polynomials
	σ_2	12	$(t^2+1)^2$
	σ_{24}	18	$(t^2-1)^2$
	$\sigma_2\pi_3'$	96	t^4-t^2+1
	$\pm\pi_3$	16,16	$(t^2\mp t+1)^2$
	$\pm\pi_3\pi_3'$	64,64	$(t\mp 1)^2(t^2\pm t+1)$
	$\pm\mathbb{I}$	1,1	$(t\mp 1)^4$

G_8	Conjugacy Classes	Size	Characteristic Polynomials
	σ_2	36	$(t^2+1)^2$
	σ_{24}	162	$(t^2-1)^2$
	$\sigma_2\pi_3'$	288	t^4-t^2+1
	$\sigma_2\pi_4'$	216	t^4+1
	$\pm\pi_3$	16,16	$(t^2\mp t+1)^2$
	$\pm\pi_3\pi_3'$	64,64	$(t\mp 1)^2(t^2\pm t+1)$
	$\pm\pi_3\pi_4'$	96,96	$t^4+t^2\mp\sqrt{2}t^3\mp\sqrt{2}t+1$
	$\pm\pi_4$	12,12	$(t^2\mp\sqrt{2}t+1)^2$
	$\pm\pi_4\pi_4'$	36,36	$(t\mp 1)^2(t^2+1)$
	$\pm\mathbb{I}$	1,1	$(t\mp 1)^4$

G_{12}	Conjugacy Classes	Size	Characteristic Polynomials
	σ_2	60	$(t^2+1)^2$
	σ_{24}	450	$(t^2-1)^2$
	$\sigma_2\pi'_5$	720	$t^4-\frac{1}{2}t^2+\frac{1}{2}\sqrt{5}t^2+1$
	$\sigma_2\pi'^2_5$	720	$t^4-\frac{1}{2}t^2-\frac{1}{2}\sqrt{5}t^2+1$
	$\sigma_2\pi'^2_5\sigma_4$	720	t^4-t^2+1
	$\pm\pi_5$	24,24	$\frac{1}{4}(2t^2\mp t\mp\sqrt{5}t+2)^2$
	$\pm\pi_5\pi'_5$	144,144	$\frac{1}{2}(t\mp 1)^2(2t^2\mp\sqrt{5}t\pm t+2)$
	$\pm\pi_5\pi'^2_5\sigma_4$	480,480	$t^4\pm\frac{1}{2}t^3\pm\frac{1}{2}\sqrt{5}t^3+\frac{1}{2}t^2$ $+\frac{1}{2}\sqrt{5}t^2\pm\frac{1}{2}t\pm\frac{1}{2}\sqrt{5}t+1$
	$\pm\pi_5^2$	24,24	$\frac{1}{4}(2t^2\pm t\mp\sqrt{5}t+2)^2$
	$\pm\pi_5^2\pi'_5$	288,288	$t^4\mp t^3+t^2\mp t+1$
	$\pm\pi_5^2\pi'^2_5$	144,144	$\frac{1}{2}(2t^2\pm t\sqrt{5}\pm t+2)(t\mp 1)^2$
	$\pm\pi_5^2\pi'^2_5\sigma_4$	480,480	$t^4\mp\frac{1}{2}t^3\pm\frac{1}{2}\sqrt{5}t^3+\frac{1}{2}t^2$ $-\frac{1}{2}\sqrt{5}t^2\mp\frac{1}{2}t\pm\frac{1}{2}\sqrt{5}t+1$
	$\pm\pi_5^2\sigma_2$	40,40	$(t^2\pm t+1)^2$
	$\pm\pi_5^2\sigma_2\pi'^2_5\sigma_4$	400,400	$(t\mp 1)^2(t^2\pm t+1)$
	$\pm\mathbb{I}$	1,1	$(t\mp 1)^4$

2.2 Poincaré series

Let $G \subseteq \mathrm{SO}(4)$ be a finite matrix group, and let $\mathbb{C}[x_0,x_1,x_2,x_3]$ denote the ring of polynomials in four variables over $\mathbb{C}$. If $g\in G$, then g acts on $f(x_0,x_1,x_2,x_3)\in\mathbb{C}[x_0,x_1,x_2,x_3]$ in the following way

$$g\cdot f(x)=f(g^{-1}x),$$

where we write $x:=(x_0,x_1,x_2,x_3)$. Let now $\mathbb{C}[x_0,x_1,x_2,x_3]_j^G$ denote the $\mathbb{C}$-vector space of homogeneous polynomials of degree j, invariant under the action of G, i.e. $\mathbb{C}[x_0,x_1,x_2,x_3]_j^G=\{\ p\in\mathbb{C}[x_0,x_1,x_2,x_3]\mid p$ is homogeneous of degree j and $p(g^{-1}x) = p(x), \forall g \in G\}$. We are interested in the dimension of this vector space. Consider the *Poincaré series*:

$$p(\mathbb{C}[x_0, x_1, x_2, x_3]^G, t) := \sum_{j=0}^{\infty} t^j \dim \mathbb{C}[x_0, x_1, x_2, x_3]^G_j .$$

Molien's theorem (cf.[5], p. 21) gives:

$$p(\mathbb{C}[x_0, x_1, x_2, x_3]^G, t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(\mathbb{I} - g^{-1}t)}. \tag{3}$$

Since $G \subseteq \mathrm{SO}(4)$, we have $\det g = \det g^{-1} = 1$, and we may write:

$$p(\mathbb{C}[x_0, x_1, x_2, x_3]^G, t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(g - \mathbb{I}t)} \tag{4}$$

where the denominator is the characteristic polynomial of g.
Remember that matrices in the same conjugacy class have the same characteristic polynomial and that the group G is disjoint union of its conjugacy classes. Let now $n_g = |[g]|$ be the order of the conjugacy class $[g]$. We write:

$$p(\mathbb{C}[x_0, x_1, x_2, x_3]^G, t) = \frac{1}{|G|} \sum_{[g] \text{ conj.class}} \frac{n_g}{\det(g - \mathbb{I}t)}.$$

Applying this result with G a bi-polyhedral group and keeping in mind the tables of the characteristic polynomials on page 21-22, we get:
$\underline{G = \mathcal{H}}$

$$p(\mathbb{C}[x_0, x_1, x_2, x_3]^{\mathcal{H}}, t) = \frac{1}{32}\left(\frac{12}{(t^2+1)^2} + \frac{18}{(t^2-1)^2} + \frac{1}{(t-1)^4} + \frac{1}{(t+1)^4}\right),$$

$\underline{G = G_6}$

$$\begin{aligned} p(\mathbb{C}[x_0, x_1, x_2, x_3]^{G_6}, t) &= \frac{1}{288}\Big(\frac{12}{(t^2+1)^2} + \frac{18}{(t^2-1)^2} + \frac{96}{(t^4-t^2+1)} \\ &+ \frac{16}{(t^2-t+1)^2} + \frac{16}{(t^2+t+1)^2} \\ &+ \frac{64}{(t-1)^2(t^2+t+1)} + \frac{64}{(t+1)^2(t^2-t+1)} \\ &+ \frac{1}{(t-1)^4} + \frac{1}{(t+1)^4}\Big), \end{aligned}$$

$\underline{G = G_8}$

$$p(\mathbb{C}[x_0,x_1,x_2,x_3]^{G_8},t) = \frac{1}{1152}\Big(\frac{36}{(t+1)^2(t^2+1)} + \frac{162}{(t-1)^2(t+1)^2} + \frac{288}{(t^4-t^2+1)} + \frac{216}{(t^4+1)} + \frac{16}{(t^2-t+1)^2} + \frac{16}{(t^2+t+1)^2} + \frac{64}{(t-1)^2(t^2+t+1)} + \frac{64}{(t+1)^2(t^2-t+1)} + \frac{96}{(t^4+t^2-\sqrt{2}t^3-\sqrt{2}t+1)} + \frac{96}{(t^4+t^2+\sqrt{2}t^3+\sqrt{2}t+1)} + \frac{12}{(t^2-\sqrt{2}t+1)^2} + \frac{12}{(t^2+\sqrt{2}t+1)^2} + \frac{36}{(t-1)^2(t^2+1)} + \frac{36}{(t^2+1)^2} + \frac{1}{(t-1)^4} + \frac{1}{(t+1)^4}\Big),$$

$\underline{G = G_{12}}$

$$p(\mathbb{C}[x_0,x_1,x_2,x_3]^{G_{12}},t) = \frac{1}{7200}\Big(\frac{60}{(t^2+1)^2} + \frac{450}{(t^2-1)^2} + \frac{720}{(t^4-\frac{1}{2}t^2+\frac{1}{2}\sqrt{5}t^2+1)} + \frac{720}{(t^4-\frac{1}{2}t^2-\frac{1}{2}\sqrt{5}t^2+1)} + \frac{720}{(t^4-t^2+1)} + \frac{24}{\frac{1}{4}(2t^2-t-\sqrt{5}t+2)^2} + \frac{24}{\frac{1}{4}(2t^2+t+\sqrt{5}t+2)^2} + \frac{144}{\frac{1}{2}(t-1)^2(2t^2-\sqrt{5}t+t+2)} + \frac{144}{\frac{1}{2}(t+1)^2(2t^2+\sqrt{5}t-t+2)} + \frac{480}{\frac{1}{2}(2t^4+t^3+\sqrt{5}t^3+t^2+\sqrt{5}t^2+t+\sqrt{5}t+2)} + \frac{480}{\frac{1}{2}(2t^4-t^3-\sqrt{5}t^3+t^2+\sqrt{5}t^2-t-\sqrt{5}t+2)}$$

$$
\begin{aligned}
&+\frac{24}{\frac{1}{4}(2t^2+t-\sqrt{5}t+2)^2}+\frac{24}{\frac{1}{4}(2t^2-t+\sqrt{5}t+2)^2}\\
&+\frac{288}{(t^4-t^3+t^2-t+1)}+\frac{288}{(t^4+t^3+t^2+t+1)}\\
&+\frac{144}{\frac{1}{2}(t-1)^2(2t^2+\sqrt{5}t+t+2)}\\
&+\frac{144}{\frac{1}{2}(t+1)^2(2t^2-\sqrt{5}t-t+2)}\\
&+\frac{480}{\frac{1}{2}(2t^4-t^3+\sqrt{5}t^3+t^2-\sqrt{5}t^2-t+\sqrt{5}t+2)}\\
&+\frac{480}{\frac{1}{2}(2t^4+t^3-\sqrt{5}t^3+t^2+\sqrt{5}t^2+t-\sqrt{5}t+2)}\\
&+\frac{40}{(t^2+t+1)^2}+\frac{40}{(t^2-t+1)^2}+\frac{400}{(t-1)^2(t^2+t+1)}\\
&+\frac{400}{(t+1)^2(t^2-t+1)}+\frac{1}{(t-1)^4}+\frac{1}{(t+1)^4}).
\end{aligned}
$$

In the following table, doing calculations with MAPLE, we give the expressions of the Poincaré series after expanding the previous polynomials as a Taylor series up to the order 14:

Group	Poincaré series
$\mathcal{H}$	$1+t^2+5t^4+6t^6+15t^8+19t^{10}+35t^{12}+44t^{14}+O(t^{16})$
G_6	$1+t^2+t^4+2t^6+3t^8+3t^{10}+7t^{12}+8t^{14}+O(t^{16})$
G_8	$1+t^2+t^4+t^6+2t^8+2t^{10}+3t^{12}+3t^{14}+O(t^{16})$
G_{12}	$1+t^2+t^4+t^6+t^8+t^{10}+2t^{12}+2t^{14}+O(t^{16})$

2.3 The spaces $\mathbb{C}[x_0, x_1, x_2, x_3]_n^{G_n}$

We defined a homogeneous polynomial p to be invariant under the action of G_n ($n = 6, 8, 12$) if $p(g^{-1}x) = p(x)$ for all $g \in G_n$. Clearly this is equivalent to p being invariant under the action of the generators of G_n. The generators of $\mathcal{H}$ give coordinate transformations:

$$x \overset{\sigma_1}{\longmapsto} \begin{pmatrix} -x_1 \\ x_0 \\ -x_3 \\ x_2 \end{pmatrix}, \quad x \overset{\sigma_2}{\longmapsto} \begin{pmatrix} -x_2 \\ x_3 \\ x_0 \\ -x_1 \end{pmatrix},$$
$$x \overset{\sigma_3}{\longmapsto} \begin{pmatrix} x_1 \\ -x_0 \\ -x_3 \\ x_2 \end{pmatrix}, \quad x \overset{\sigma_4}{\longmapsto} \begin{pmatrix} x_2 \\ x_3 \\ -x_0 \\ -x_1 \end{pmatrix},$$

it is a well known fact that the invariant polynomials have even degree only. Since $\mathcal{H} \subseteq G_n$, also under these groups we have invariant polynomials just in even degree (as the table above shows). If the dimension of $\mathbb{C}[x_0, x_1, x_2, x_3]_j^{G_n}$ is one, then we have just one generator. It is the multiple complex quadric $Q_j(x) := (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{\frac{j}{2}}$. In degree $n = 6, 8, 12$, we find $\dim \mathbb{C}[x_0, x_1, x_2, x_3]_n^{G_n} = 2$, so we have two generators, the quadric and another homogeneous polynomial. This explains, in particular, why we used the notation G_n: the first non trivial invariant polynomial appears in degree n. We give now a method to calculate it. We start with a basis $p_1^{(n)}, \ldots, p_m^{(n)}$ of $\mathbb{C}[x_0, x_1, x_2, x_3]_n^{\mathcal{H}}$ (cf. e.g. [14], [16]). Then, considering a linear combination $F_n(x) := A_1 p_1^{(n)} + \ldots + A_m p_m^{(n)}$ $(A_i \in \mathbb{C})$, we impose the condition $F_n(\sigma x) = F_n(x)$ for all the generators σ in $G_n \backslash \mathcal{H}$. In this way we can find a basis of $\mathbb{C}[x_0, x_1, x_2, x_3]_n^{G_n}$.
In degree 6 and 8 the calculations are not difficult if left to MAPLE. In degree 12 it is however more difficult, because of the relatively high number of parameters in the general expression of $F_{12}(x)$. We show how to find the invariant polynomial of degree 6 and 8 using this method. In degree 12 we give first the expression of the invariant polynomial as a linear combination of some symmetric polynomials and in the generators of $\mathbb{C}[x_0, x_1, x_2, x_3]_{12}^{\mathcal{H}}$. Then at the end of the chapter we give it explicitly in the x_i's.

2.3.1 Degree 6

We have $\dim \mathbb{C}[x_0, x_1, x_2, x_3]_6^{\mathcal{H}} = 6$ and a basis for this vector space is given by the polynomials

$$\begin{aligned}
p_1 &:= x_0^6 + x_1^6 + x_2^6 + x_3^6, \\
p_2 &:= x_0^2 x_1^2 x_3^2 + x_0^2 x_1^2 x_2^2 + x_0^2 x_2^2 x_3^2 + x_1^2 x_2^2 x_3^2, \\
p_3 &:= x_0^4 x_1^2 + x_1^4 x_0^2 + x_2^4 x_3^2 + x_3^4 x_2^2, \\
p_4 &:= x_0^4 x_2^2 + x_2^4 x_0^2 + x_1^4 x_3^2 + x_3^4 x_1^2,
\end{aligned}$$

$$\begin{aligned} p_5 &:= x_0^4x_3^2+x_3^4x_0^2+x_1^4x_2^2+x_2^4x_1^2,\\ p_6 &:= x_0^3x_1x_2x_3+x_0x_1^3x_2x_3+x_0x_1x_2^3x_3+x_0x_1x_2x_3^3. \end{aligned}$$

We consider now the group G_6. Its generators $\pi_3, \pi_3' \in G_6\backslash\mathcal{H}$ give the following transformations:

$$x \overset{\pi_3}{\longmapsto} \frac{1}{2}\begin{pmatrix} x_0-x_1+x_2-x_3\\ x_0+x_1-x_2-x_3\\ -x_0+x_1+x_2-x_3\\ x_0+x_1+x_2+x_3 \end{pmatrix}, \quad x \overset{\pi_3'}{\longmapsto} \frac{1}{2}\begin{pmatrix} x_0+x_1-x_2+x_3\\ -x_0+x_1-x_2+x_3\\ x_0+x_1+x_2-x_3\\ -x_0+x_1+x_2+x_3 \end{pmatrix}$$

Writing $F_6(x) := A_1p_1 + A_2p_2 + \ldots + A_6p_6$ and imposing the condition $F_6(\pi_3 x) = F_6(x)$, $F_6(\pi_3' x) = F_6(x)$. We get:

$$A_2 = 15A_1 - 3A_3,\ A_3 = A_4 = A_5 \text{ and } A_6 = 0$$

Then $F_6(x) = \alpha\underbrace{(p_1+15p_2)}_{:=f_1} + \beta\underbrace{(-3p_2+p_3+p_4+p_5)}_{:=f_2}$. Now the sum $f_1 + 3f_2 = Q_6(x)$ and with

$$S_6(x) := f_1 = x_0^6+x_1^6+x_2^6+x_3^6+15(x_0^2x_1^2x_3^2+x_0^2x_1^2x_2^2+x_0^2x_2^2x_3^2+x_1^2x_2^2x_3^2)$$

we write an element of $\mathbb{C}[x_0,x_1,x_2,x_3]_6^{G_6}$ as

$$\lambda Q_6(x) + \mu S_6(x), \qquad \lambda, \mu \in \mathbb{C}.$$

2.3.2 Degree 8

In this case we have $\dim \mathbb{C}[x_0,x_1,x_2,x_3]_8^{\mathcal{H}} = 15$ and a basis is given by:

$$\begin{aligned} q_1 &:= x_0^8+x_1^8+x_2^8+x_3^8,\\ q_2 &:= x_0^4x_1^4+x_2^4x_3^4,\\ q_3 &:= x_0^4x_2^4+x_1^4x_3^4,\\ q_4 &:= x_0^4x_3^4+x_1^4x_2^4,\\ q_5 &:= x_0^2x_1^2x_2^2x_3^2,\\ q_6 &:= x_0^3x_1^3x_2x_3+x_0x_1x_2^3x_3^3,\\ q_7 &:= x_0^3x_2^3x_1x_3+x_0x_2x_1^3x_3^3,\\ q_8 &:= x_0^3x_3^3x_2x_1+x_0x_3x_2^3x_1^3,\\ q_9 &:= x_0^6x_1^2+x_1^6x_0^2+x_2^6x_3^2+x_3^6x_2^2,\\ q_{10} &:= x_0^6x_2^2+x_2^6x_0^2+x_1^6x_3^2+x_3^6x_1^2,\\ q_{11} &:= x_0^6x_3^2+x_3^6x_0^2+x_1^6x_2^2+x_2^6x_1^2, \end{aligned}$$

$$\begin{aligned}
q_{12} &:= x_0^4x_1^2x_2^2 + x_1^4x_0^2x_3^2 + x_3^4x_2^2x_1^2 + x_2^4x_3^2x_0^2,\\
q_{13} &:= x_0^4x_1^2x_3^2 + x_1^4x_0^2x_2^2 + x_3^4x_2^2x_0^2 + x_2^4x_3^2x_1^2,\\
q_{14} &:= x_0^4x_2^2x_3^2 + x_2^4x_0^2x_1^2 + x_3^4x_0^2x_1^2 + x_1^4x_3^2x_2^2,\\
q_{15} &:= x_0^5x_1x_2x_3 + x_0x_1^5x_2x_3 + x_0x_1x_2^5x_3 + x_0x_1x_2x_3^5.
\end{aligned}$$

We write an element $F_8(x) = A_1q_1 + \ldots + A_{15}q_{15}$ and we apply the same method as before. Since $G_6 \subseteq G_8$ we consider first the space $\mathbb{C}[x_0, x_1, x_2, x_3]_8^{G_6}$. After calculations with MAPLE, we find generators

$$\begin{aligned}
g_1 &:= -24q_5 + q_{11} + q_{12} + q_9 + q_{10} + q_{13} + q_{14},\\
g_2 &:= q_1 + 84q_5 + 14q_{12} + 14q_{13} + 14q_{14},\\
g_3 &:= q_2 + q_3 + q_4 - 18q_5 + q_9 + q_{10} + q_{11}.
\end{aligned}$$

So we may write the element $F_8(x)$ in $\mathbb{C}[x_0, x_1, x_2, x_3]_8^{G_6}$ as a combination: $F_8(x) = B_1g_1 + B_2g_2 + B_3g_3$, $B_i \in \mathbb{C}$.
We impose now the conditions $F_8(\pi_4x) = F_8(x)$, $F_8(\pi_4'x) = F_8(x)$, i.e. the transformations:

$$x \xrightarrow{\pi_4} \frac{1}{\sqrt{2}}\begin{pmatrix} x_0 - x_1\\ x_0 + x_1\\ x_2 - x_3\\ x_2 + x_3 \end{pmatrix}, \quad x \xrightarrow{\pi_4'} \frac{1}{\sqrt{2}}\begin{pmatrix} x_0 + x_1\\ -x_0 + x_1\\ x_2 - x_3\\ x_2 + x_3 \end{pmatrix}.$$

The dimension of $\mathbb{C}[x_0, x_1, x_2, x_3]_8^{G_8}$ is two and we find the two invariant polynomials:

$$G_1 := 7g_1 + g_2 \text{ and } G_2 := -3g_1 + 2g_3$$

Observe that $Q_8(x) = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^4 = G_1 + 3G_2$. We put

$$\begin{aligned}
S_8(x) &:= G_1 + 7G_2 = 7g_1 + g_2 - 21g_1 + 14g_3 = g_2 + 14(g_3 - g_1)\\
&= x_0^8 + x_1^8 + x_2^8 + x_3^8 + 14(x_0^4x_1^4 + x_0^4x_2^4 + x_0^4x_3^4 + x_1^4x_2^4 + x_1^4x_3^4 + x_2^4x_3^4)\\
&\quad +168x_0^2x_1^2x_2^2x_3^2.
\end{aligned}$$

We write an element in $\mathbb{C}[x_0, x_1, x_2, x_3]_8^{G_8}$ as

$$\lambda Q_8(x) + \mu S_8(x), \qquad \lambda, \mu \in \mathbb{C}.$$

2.3.3 Degree 12

In this case $\dim \mathbb{C}[x_0, x_1, x_2, x_3]_{12}^{\mathcal{H}} = 35$, the generators are obtained as products of the five invariant polynomials in degree four:

$$\begin{aligned} l_1 &:= x_0^4 + x_1^4 + x_2^4 + x_3^4, \\ l_2 &:= x_0^2x_1^2 + x_2^2x_3^2, \\ l_3 &:= x_0^2x_2^2 + x_1^2x_3^2, \\ l_4 &:= x_0^2x_3^2 + x_1^2x_2^2, \\ l_5 &:= x_0x_1x_2x_3. \end{aligned}$$

Considering a general linear combination of the 35 invariants and proceeding as before one can find the equation of the non trivial invariant polynomial in $\mathbb{C}[x_0, x_1, x_2, x_3]_{12}^{G_{12}}$. We call it $S_{12}(x)$. We give its expression in terms of some symmetric polynomials. Abbreviate $q_i := x_i^2$.

$$\begin{aligned} S_{51} &:= q_0^5(q_1+q_2+q_3) + q_1^5(q_0+q_2+q_3) + q_2^5(q_0+q_1+q_3) \\ &\quad +q_3^5(q_0+q_1+q_2), \\ S_{42} &:= q_0^4(q_1^2+q_2^2+q_3^2) + q_1^4(q_0^2+q_2^2+q_3^2) + q_2^4(q_0^2+q_1^2+q_3^2) \\ &\quad +q_3^4(q_0^2+q_1^2+q_2^2), \\ S_{411} &:= q_0^4(q_1q_2+q_1q_3+q_2q_3) + q_1^4(q_0q_2+q_0q_3+q_2q_3) \\ &\quad +q_2^4(q_0q_1+q_0q_3+q_1q_3) + q_3^4(q_0q_1+q_0q_2+q_1q_2), \\ S_{33} &:= q_0^3(q_1^3+q_2^3+q_3^3) + q_1^3(q_2^3+q_3^3) + q_2^3q_3^3, \\ S_{321} &:= q_0^3(q_1^2(q_2+q_3) + q_2^2(q_1+q_3) + q_3^2(q_1+q_2)) + q_1^3(q_0^2(q_2+q_3) \\ &\quad +q_2^2(q_0+q_3) + q_3^2(q_0+q_2)) + q_2^3(q_0^2(q_1+q_3) + q_1^2(q_0+q_3) \\ &\quad +q_3^2(q_0+q_1)) + q_3^3(q_0^2(q_1+q_2) + q_1^2(q_0+q_2) + q_2^2(q_0+q_1)), \\ S_{3111} &:= q_0q_1q_2q_3(q_0^2+q_1^2+q_2^2+q_3^2), \\ S_{222} &:= (q_0q_1q_2)^2 + (q_0q_1q_3)^2 + (q_0q_2q_3)^2 + (q_1q_2q_3)^2, \\ S_{2211} &:= q_0q_1q_2q_3(q_0(q_1+q_2+q_3) + q_1(q_2+q_3) + q_2q_3). \end{aligned}$$

We say that a polynomial $p := p(x_0, x_1, x_2, x_3)$ is *totally symmetric* if it is invariant under each coordinate permutation. Similarly we say that p is *anti-symmetric* if it is invariant under each even coordinate permutation $\sigma \in A_4$ and for $\gamma \in S_4 \backslash A_4$ holds

$$p(\gamma(x_0, x_1, x_2, x_3)) = -p(x_0, x_1, x_2, x_3).$$

With this terminology the totally symmetric part of the invariant $S_{12}(x)$ is given by

$$f_s := 2S_{51} - 6S_{42} - 12S_{411} + 14S_{33} + 9S_{321} + 348S_{3111} + 30S_{222} - 270S_{2211}.$$

and the anti-symmetric part is

$$33\sqrt{5}f_a$$

with

$$\begin{aligned} f_a &:= q_0^3(q_1^2q_2 - q_1q_2^2 + q_2^2q_3 - q_2q_3^2 + q_3^2q_1 - q_3q_1^2) - q_1^3(q_2^2q_3 - q_2q_3^2 \\ &\quad + q_3^2q_0 - q_3q_0^2 + q_0^2q_2 - q_0q_2^2) + q_2^3(q_0^2q_1 - q_0q_1^2 + q_1^2q_3 - q_1q_3^2 \\ &\quad + q_3^2q_0 - q_3q_0^2) - q_3^3(q_0^2q_1 - q_0q_1^2 + q_1^2q_2 - q_1q_2^2 + q_2^2q_0 - q_2q_0^2). \end{aligned}$$

In conclusion:

$$S_{12}(x) := f_s + 33\sqrt{5}f_a$$

This invariant polynomial can be expressed in the $l_i's$ powers. In fact putting

$$\begin{array}{llllll} s_{1,0} & := & l_1l_2l_3 + l_1l_3l_4 + l_1l_2l_4, & s_{1,1} & := & l_1^2l_2 + l_1^2l_3 + l_1^2l_4, \\ s_{1,2} & := & l_1l_2^2 + l_1l_3^2 + l_1l_4^2, & s_{5,1} & := & l_5^2l_2 + l_5^2l_3 + l_5^2l_4, \\ s_{2,3}^- & := & l_2^2l_3 - l_3^2l_2, & s_{3,4}^- & := & l_3^2l_4 - l_4^2l_3, \\ s_{4,2}^- & := & l_4^2l_2 - l_2^2l_4, & s_{2,3}^+ & := & l_2^2l_3 + l_3^2l_2, \\ s_{3,4}^+ & := & l_3^2l_4 + l_4^2l_3, & s_{4,2}^+ & := & l_4^2l_2 + l_2^2l_4, \\ s_{2,3,4} & := & l_2^3 + l_3^3 + l_4^3, & & & \end{array}$$

the equation has the form:

$$\begin{aligned} S_{12}(x) &:= 33\sqrt{5}(s_{2,3}^- + s_{3,4}^- + s_{4,2}^-) + 19(s_{2,3}^+ + s_{3,4}^+ + s_{4,2}^+) + 10s_{2,3,4} \\ &\quad -14s_{1,0} + 2s_{1,1} - 6s_{1,2} - 352s_{5,1} + 336l_5^2l_1 + 48l_2l_3l_4. \end{aligned}$$

This shows that $S_{12}(x)$ is in fact invariant under the action of $\mathcal{H}$. The generators π_5,π_5' correspond to the transformations

$$x \stackrel{\pi_5}{\longmapsto} \frac{1}{2}\begin{pmatrix} \tau x_0 + (1-\tau)x_2 - x_3 \\ \tau x_1 - x_2 + (\tau-1)x_3 \\ (\tau-1)x_0 + x_1 + \tau x_2 \\ x_0 + (1-\tau)x_1 + \tau x_3 \end{pmatrix}, \quad x \stackrel{\pi_5'}{\longmapsto} \frac{1}{2}\begin{pmatrix} \tau x_0 + (\tau-1)x_2 + x_3 \\ \tau x_1 - x_2 + (\tau-1)x_3 \\ (1-\tau)x_0 + x_1 + \tau x_2 \\ -x_0 + (1-\tau)x_1 + \tau x_3 \end{pmatrix}.$$

Applying these coordinate transformations to $S_{12}(x)$ and doing the calculations with MAPLE, one sees that $S_{12}(\pi_5 x) = S_{12}(x)$ and $S_{12}(\pi_5' x) = S_{12}(x)$. We write an element of $\mathbb{C}[x_0, x_1, x_2, x_3]_{12}^{G_{12}}$ as

$$\lambda Q_{12}(x) + \mu S_{12}(x), \qquad \lambda, \mu \in \mathbb{C}.$$

Before give the explicit expression of the invariant polynomial $S_{12}(x)$, we give the following

Proposition 2.3 *For* $n = 6, 8, 12$, *we have* $S_n(x) \neq \lambda Q_n(x)$, *for all* $\lambda \in \mathbb{C}$. *Moreover the polynomial* $Q_2(x)$ *does not divide* $S_n(x)$.

Proof. We show that for some point $p \in \mathbb{C}^4$, we have $Q_2(p) = 0$ but $S_n(p) \neq 0$. Consider $p = (i\sqrt{2}, 1, 1, 0)$ then

$$Q_2(p) = (i\sqrt{2})^2 + 1 + 1 = 0$$

On the other hand we have

$$\begin{aligned}
S_6(p) &= (i\sqrt{2})^6 + 1 + 1 + 15(i\sqrt{2})^2 \\
&= (-2)^3 + 1 + 1 + 15(-2) \\
&= -36, \\
S_8(p) &= (i\sqrt{2})^8 + 1 + 1 + 14(2(i\sqrt{2})^4 + 1) \\
&= (-2)^4 + 1 + 1 + 14(2(-2)^2 + 1) \\
&= 144,
\end{aligned}$$

and a calculation with MAPLE shows $S_{12}(p) = -726$. □

The invariant polynomial $S_{12}(x)$

$$\begin{aligned}
&2x_0^{10}x_1^2 + 2x_0^{10}x_2^2 + 2x_0^{10}x_3^2 + 2x_1^{10}x_0^2 + 2x_1^{10}x_2^2 + 2x_1^{10}x_3^2 \\
&+2x_2^{10}x_0^2 + 2x_2^{10}x_1^2 + 2x_2^{10}x_3^2 + 2x_3^{10}x_0^2 + 2x_3^{10}x_1^2 + 2x_3^{10}x_2^2 \\
&-6x_0^8x_1^4 - 6x_0^8x_2^4 - 6x_0^8x_3^4 - 6x_1^8x_0^4 - 6x_1^8x_2^4 - 6x_1^8x_3^4 \\
&-6x_2^8x_0^4 - 6x_2^8x_1^4 - 6x_2^8x_3^4 - 6x_3^8x_0^4 - 6x_3^8x_1^4 - 6x_3^8x_2^4 \\
&-12x_0^8x_1^2x_2^2 - 12x_0^8x_1^2x_3^2 - 12x_0^8x_2^2x_3^2 - 12x_1^8x_0^2x_2^2 - 12x_1^8x_0^2x_3^2 - 12x_1^8x_2^2x_3^2 \\
&-12x_2^8x_0^2x_1^2 - 12x_2^8x_0^2x_3^2 - 12x_2^8x_1^2x_3^2 - 12x_3^8x_0^2x_1^2 - 12x_3^8x_0^2x_2^2 - 12x_3^8x_1^2x_2^2 \\
&+14x_0^6x_1^6 + 14x_0^6x_2^6 + 14x_0^6x_3^6 + 14x_1^6x_2^6 + 14x_1^6x_3^6 + 14x_2^6x_3^6 \\
&+9x_0^6x_1^4x_2^2 + 9x_0^6x_1^4x_3^2 + 9x_0^6x_2^4x_1^2 + 9x_0^6x_2^4x_3^2 + 9x_0^6x_3^4x_1^2 + 9x_0^6x_3^4x_2^2 \\
&+9x_1^6x_0^4x_2^2 + 9x_1^6x_0^4x_3^2 + 9x_1^6x_2^4x_0^2 + 9x_1^6x_2^4x_3^2 + 9x_1^6x_3^4x_0^2 + 9x_1^6x_3^4x_2^2 \\
&+9x_2^6x_0^4x_1^2 + 9x_2^6x_0^4x_3^2 + 9x_2^6x_1^4x_0^2 + 9x_2^6x_1^4x_3^2 + 9x_2^6x_3^4x_0^2 + 9x_2^6x_3^4x_1^2 \\
&+9x_3^6x_0^4x_1^2 + 9x_3^6x_0^4x_2^2 + 9x_3^6x_1^4x_0^2 + 9x_3^6x_1^4x_2^2 + 9x_3^6x_2^4x_0^2 + 9x_3^6x_2^4x_1^2 \\
&+348x_0^6x_1^2x_2^2x_3^2 + 348x_1^6x_0^2x_2^2x_3^2 + 348x_2^6x_0^2x_1^2x_3^2 + 348x_3^6x_0^2x_1^2x_2^2 \\
&+30x_0^4x_1^4x_2^4 + 30x_0^4x_1^4x_3^4 + 30x_0^4x_2^4x_3^4 + 30x_1^4x_2^4x_3^4 \\
&-270x_0^4x_1^4x_2^2x_3^2 - 270x_0^4x_2^4x_1^2x_3^2 - 270x_0^4x_3^4x_1^2x_2^2 \\
&-270x_1^4x_2^4x_0^2x_3^2 - 270x_1^4x_3^4x_0^2x_2^2 - 270x_2^4x_3^4x_0^2x_1^2 \\
&+33\sqrt{5}(x_0^6x_1^4x_2^2 - x_0^6x_1^2x_2^4 + x_0^6x_2^4x_3^2 - x_0^6x_2^2x_3^4) \\
&+33\sqrt{5}(x_0^6x_3^4x_1^2 - x_0^6x_3^2x_1^4 + x_1^6x_0^4x_3^2 - x_1^6x_0^2x_3^4) \\
&+33\sqrt{5}(x_1^6x_2^4x_0^2 - x_1^6x_2^2x_0^4 + x_1^6x_3^4x_2^2 - x_1^6x_3^2x_2^4) \\
&+33\sqrt{5}(x_2^6x_0^4x_1^2 - x_2^6x_0^2x_1^4 + x_2^6x_1^4x_3^2 - x_2^6x_1^2x_3^4) \\
&+33\sqrt{5}(x_2^6x_3^4x_0^2 - x_2^6x_3^2x_0^4 + x_3^6x_0^4x_2^2 - x_3^6x_0^2x_2^4) \\
&+33\sqrt{5}(x_3^6x_1^4x_0^2 - x_3^6x_1^2x_0^4 + x_3^6x_2^4x_1^2 - x_3^6x_2^2x_1^4).
\end{aligned}$$

3 Pencils of surfaces

A homogeneous polynomial $f(x) \in \mathbb{C}[x_0, x_1, x_2, x_3]$ defines a surface $S := \{f(x) = 0\}$ in three dimensional complex projective space $\mathbb{P}^3_{\mathbb{C}}$. If f is invariant under the action of the group G_n ($n = 6, 8, 12$), the surface S is symmetric. The two invariant polynomials in degree 6, 8 and 12 which we found at the end of chapter 2, define pencils of surfaces in $\mathbb{P}^3_{\mathbb{C}}$. We have seen that in degree 2 we have just one invariant, the quadric Q_2: $\{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\}$, or equivalently in $\mathbb{C}^3$ the complex sphere $x^2 + y^2 + z^2 = -1$ which has, of course, "any" kind of symmetries. The equations of the pencils are given by:

$$F_n(\lambda) : S_n(x) + \lambda Q_n(x) = 0, \qquad \lambda \in \mathbb{P}^1$$

For $\lambda = 0$ we get the surface $\{S_n(x) = 0\}$ and for $\lambda = \infty$ we get the multiple quadric $\{Q_n(x) = 0\}$. In this section, after some general results, we calculate the base locus of the pencils, i.e. the set of points $p \in \mathbb{P}^3_{\mathbb{C}}$ s.t. $S_n(p) + \lambda Q_n(p) = 0$ for all $\lambda \in \mathbb{P}^1$. For simplicity we denote the surfaces $\{S_n(x) = 0\}$ and $\{Q_n(x) = 0\}$, $n = 6, 8, 12$, by S_n and Q_n.
Observe that the pencils are invariant under the action of bigger groups, more precisely, let C denote the matrix given on page 17, then

Lemma 3.1 **(i)** *The group* $< G_n, C >$ *has order* $2 \cdot |G_n|$, $n = 6, 8, 12$. *Explicitly*

$$| < G_6, C > | = 576, \quad | < G_8, C > | = 2304, \quad | < G_{12}, C > | = 14400.$$

(ii) *The surfaces of the pencil* $F_n(\lambda)$ *are invariant under the action of* $< G_n, C >$.

Proof. **(i)**The set $C \cdot G_n = \{C \cdot g |\ g \in G_n\}$ is contained in $< G_n, C >$ and has $|G_n|$ elements. Moreover $C^2 = \mathbb{I}$ and $C\sigma(p,q)C = \sigma(q,p) \in G_n$, therefore $| < G_n, C > | = 2 \cdot |G_n|$.
(ii) The matrix C maps a point $(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3_{\mathbb{C}}$ to $(x_0 : -x_1 : -x_2 : -x_3)$. The surfaces in $F_n(\lambda)$ are defined by polynomials in the x_i's squares, therefore are invariant under this coordinate transformation. □

We do, now, some identification. Define

$$\mathbb{H}_{\mathbb{C}} = \{x_0 q_0 + x_1 q_1 + x_2 q_2 + x_3 q_3 |\ (x_0, x_1, x_2, x_3) \in \mathbb{C}^4\},$$

then $\mathbb{P}(\mathbb{H}_{\mathbb{C}}) \cong \mathbb{P}^3_{\mathbb{C}}$. Hence in the basis $q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $q_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $q_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $q_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, we can write a point $x = (x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3_{\mathbb{C}}$

as

$$x = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}.$$

We have $\det(x) = x_0^2 + x_1^2 + x_2^2 + x_3^2$, if $\det(x) = 0$, then $x \in Q_2$, hence similarly to [22], we identify $\mathbb{P}^3_{\mathbb{C}} \backslash Q_2$ with $\mathbb{P}\mathrm{GL}(2)$, the projectivization of the space of invertible complex 2×2-matrices. We can explain this identification in another way. Consider the Segre embedding, which is the analogous in the projective case of the map (2) on page 16,

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \longrightarrow & \mathbb{P}^3_{\mathbb{C}} \\ ((a_0 : a_1), (b_0 : b_1)) & \longmapsto & (a_0 b_0 : a_0 b_1 : a_1 b_0 : a_1 b_1). \end{array}$$

The points of $\mathbb{P}^1 \times \mathbb{P}^1$ satisfy

$$\det \begin{pmatrix} a_0 b_0 & a_0 b_1 \\ a_1 b_0 & a_1 b_1 \end{pmatrix} = 0.$$

So they correspond to the rank one matrices in the projectivization of the space of 2×2-complex matrices. The complement $\mathbb{P}^3_{\mathbb{C}} \backslash \{\mathbb{P}^1 \times \mathbb{P}^1\}$ is identified with $\mathbb{P}\mathrm{GL}(2)$.
Often we will identify a point of $\mathbb{P}^3_{\mathbb{C}}$ with the corresponding matrix without mentioning it.

Definition 3.1 Let $G \subseteq \mathbb{P}\mathrm{GL}(4, \mathbb{C})$ be a group acting on $\mathbb{P}^3_{\mathbb{C}}$. Then $z \in \mathbb{P}^3_{\mathbb{C}}$ is called a *fix point* if there is a $\sigma \in G$ ($\sigma \neq \pm\, \mathbb{I}$), s.t. $\sigma z = z$. We call

$$\mathrm{Fix}(z) = \mathrm{Fix}_G(z) := \{g \in G \mid gz = z\} \subseteq G$$

the *fix group* of z and

$$O(z) = O_G(z) := \{gz \mid g \in G\} \subseteq \mathbb{P}^3_{\mathbb{C}}$$

the *orbit* of z. We have the formula:

$$|\mathrm{Fix}(z)| \cdot |O(z)| = |G| \tag{5}$$

Definition 3.2 We call a line $L \subseteq \mathbb{P}^3_{\mathbb{C}}$, a *line of fix points* (or *fix line*) of an element $\sigma \in G_n$ $(n = 6, 8, 12)$ if for every $x \in L$ holds $\sigma x = x$

Proposition 3.1 *The matrices $\sigma(p, \mathbb{I}), \sigma(\mathbb{I}, q) \in G_n$, have in $\mathbb{P}^3_{\mathbb{C}}$ two disjoint lines of fix points each. These are contained on the quadric Q_2 and belong to one ruling, respectively to the other ruling of Q_2.*

Proof. Using the lemma 2.2 we see that the matrices $\sigma(p,\mathbb{I}), \sigma(\mathbb{I},q)$ have two eigenvalues with multiplicity two each. The eigenspaces are lines of $\mathbb{P}^3_{\mathbb{C}}$ and these are spanned by points, which correspond to matrices of rank one (cf. lemma 2.2). By the identification given by the map (2) on page 16, it follows that the fix lines of $\sigma(p,\mathbb{I})$ are lines of the ruling $\{v\} \times \mathbb{P}^1$ and the fix lines of $\sigma(\mathbb{I},q)$ are lines of the ruling $\mathbb{P}^1 \times \{w\}$, v, $w \in \mathbb{P}^1$. □

3.1 Base locus

Definition 3.3 The *base locus* of the pencil $F_n(\lambda)$, $n = 6, 8, 12$ is the variety

$$\{x \in \mathbb{P}^3_{\mathbb{C}} \mid S_n(x) + \lambda Q_n(x) \equiv 0 \text{ for all } \lambda \in \mathbb{P}^1\}.$$

Observe that if a point p is in the base locus, in particular $S_n(p) = 0$ and $Q_n(p) = 0$. On the other hand the points $x \in \mathbb{P}^3_{\mathbb{C}}$ s.t. $S_n(x)= Q_n(x) = 0$ are in the base locus. Hence the base locus is the intersection $S_n \cap Q_n$. In particular it is invariant under the action of G_n.
Since Q_n is a multiple quadric, $Q_n(x) = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{\frac{n}{2}}$, the base locus is not reduced. Define $\mathcal{B}_n := Q_2 \cap S_n$.
We consider now the groups $\sigma(\tilde{G},\mathbb{I})$ and $\sigma(\mathbb{I},\tilde{G})$, ($\tilde{G}= \tilde{A}_4$, $\tilde{S}_4$, $\tilde{A}_5$) which modulo $\{\pm\mathbb{I}\}$ are isomorphic to the subgroups T, O and $I \subseteq \mathrm{SO}(3)$. It is a well known fact that under the action of these groups there are orbits of the following lengths,

tetrahedron	octahedron	icosahedron
12, 6, 4	24, 12, 8, 6	60, 30, 20, 12

Moreover, observe that
i) the group $\sigma(\tilde{G},\mathbb{I})$ acts on the lines of the first ruling $\{v\} \times \mathbb{P}^1$ and lets invariant each line of the second ruling $\mathbb{P}^1 \times \{w\}$. Vice versa $\sigma(\mathbb{I},\tilde{G})$ acts on $\mathbb{P}^1 \times \{w\}$ and lets invariant each line of $\{v\} \times \mathbb{P}^1$.
ii) Denote by $\mathcal{L}_n$, $\mathcal{L}'_n$ the sets of lines in $\{v\} \times \mathbb{P}^1$, resp. $\mathbb{P}^1 \times \{w\}$ of the orbit of length n. The matrix C maps lines of $\mathcal{L}_n$ to lines of $\mathcal{L}'_n$.
Using these facts we show
1) the variety $\mathcal{B}_n$ is reduced, i.e. does not contain multiple components.
2) The base locus splits in $2n$ lines, n of each ruling of Q_2.

Proof of **1)**. By Bezout's theorem $\deg(Q_2 \cap S_n) = 2n$. If $Q_2 \cap S_n$ is not reduced then there is a component $V \subseteq Q_2 \cap S_n$ s.t. Q_2 and S_n meet with multiplicity at least two, this is the case when V is singular on S_n or S_n and Q_2 are tangent at V. Consider a line L of one of the two rulings, not in $\mathcal{B}_n$, and which meets V in at least one point. W.l.o.g. assume L in the ruling $\{v\}\times\mathbb{P}^1$. Let $x \in L \cap V$. We have $\text{mult}_x(L \cdot S_n) \geq 2$. The group $\sigma(\mathbb{I}, \tilde{G})$ acts on L, so we consider the orbit of x under this group. By the table on page 35, we see that L and S_n meet at more then n points computed with multiplicity, so $L \subseteq S_n$. This contradicts the assumption. We have shown that $Q_2 \cap S_n$ is reduced. □

Proof of **2)**. Take a line L of the first or of the second ruling $L \not\subseteq \mathcal{B}_n$. The curve $\mathcal{B}_n$ has bi-degree (n, n) on Q_2, so $|L \cap \mathcal{B}_n| = n$. W.l.o.g. assume that L is in the ruling $\{v\} \times \mathbb{P}^1$. The group $\sigma(\mathbb{I}, \tilde{G})$ acts on the points of L. Let $x \in L \cap \mathcal{B}_n$, then by the table on page 35, the orbit of x under $\sigma(\mathbb{I}, \tilde{G})$ must have length n. Hence x belongs to a line of $\mathcal{L}'_n$. As we have infinitely many lines like L, the lines $\mathcal{L}'_n$ are contained in $\mathcal{B}_n$. By **ii)** above and lemma 3.1 the lines in $\mathcal{L}_n$ are contained in $\mathcal{B}_n$ too. □

4 Singular surfaces

In this chapter we show that the general surface in the pencil $F_n(\lambda)$ ($n = 6, 8, 12$) is smooth and we find the singular ones. After some general considerations on the singular points of the pencils, we show that each pencil contains exactly four singular surfaces, which have nodes and no further singularities.

Lemma 4.1 *Let p be a singular point on a surface of the pencil $F_n(\lambda)$ (not on Q_n), then p is not contained in the complex quadric.*

Proof. Let p be a singular point on the surface $F_n(\lambda_0)$: $S_n(x) + \lambda_0 Q_n(x) = 0$ and assume that $p \in Q_2$. We have

$$\partial_i S_n(p) + \lambda_0 \partial_i Q_n(p) = 0 \text{ for all } i = 0, 1, 2, 3. \tag{6}$$

Since $Q_n(p) = 0$ we get $S_n(p) = 0$ too. Moreover since $\partial_i Q_n(x) = n x_i (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{\frac{n}{2}-1}$, we get $\partial_i Q_n(p) = 0$ too. The equality (6) gives $\partial_i S_n(p) = 0$ for all $i = 0, 1, 2, 3$. This shows that p is a singular point of $Q_2 \cap S_n$. This consists of $2n$ lines which meet each other at n^2 points. Hence p must be an intersection point of two lines. If S_n is singular at p it follows that S_n is singular at all the n^2 points of intersection of the lines in the base locus, in fact they form one orbit under the action of $\sigma(\tilde{G}, \mathbb{I})$ and $\sigma(\mathbb{I}, \tilde{G})$ (notation of chapter 3). In particular S_n has n singular points on a line L in $Q_2 \cap S_n$. A hypersurface $\partial_i S_n = 0$ ($i = 0, 1, 2, 3$) has degree $n - 1$, therefore it intersects L in $n - 1$ points. So S_n has at most $n - 1$ singular points on L. It follows that L is singular on S_n. Hence S_n and Q_2 meet at L and so at all the $2n$ lines of $Q_2 \cap S_n$ with multiplicity at least 2. This is not possible, in fact $\deg(Q_2 \cap S_n) = 2n$. This shows that $p \notin Q_2$. □

Proposition 4.1 *The general surface in the pencil $F_n(\lambda)$ is smooth.*

Proof. The general surface in the pencil is smooth away from the base locus (Bertini's theorem, cf. e.g. [12], p.137). The base locus is the set $Q_2 \cap S_n \subseteq Q_2$. By lemma 4.1 we have no singularities on Q_2.

Proposition 4.2 *A surface $S \in F_n(\lambda)$ (not Q_n) has only isolated singularities.*

Proof. Assume that $S := \{Q_n(x) + \lambda_0 S_n(x) = 0\}$ contains a singular curve. This meets Q_2 in at least one point p, which is singular on S. By the lemma 4.1, this is not possible. □

Resuming the results about the surfaces of the pencils $F_n(\lambda)$:

- the general surface is smooth,
- the surfaces in the pencils, different from Q_n, are irreducible and reduced,
- the singular ones (not Q_n) have only isolated singularities.

Proposition 4.3 *A singular point on a surface in the pencil $F_n(\lambda)$, $n = 6, 8, 12$, is a fix point under G_n in sense of definition 3.1. Moreover, as vector of $\mathbb{C}^4$, it is eigenvector of a matrix with eigenvalue $+1$ or -1.*

Proof. It is possible to obtain a first rough estimate of the maximal number of singular points on a surface S of degree n in $\mathbb{P}^3_{\mathbb{C}}$ in the following way. If S has equation $\{F = 0\}$ then a point on S is singular if and only if $\partial_i F(p) = 0$ for all $i = 0, 1, 2, 3$. The singular points on S are solutions of $\{\partial_i F(p) = 0, i = 0, 1, 2, 3\}$. These are equations of hypersurfaces of degree $n - 1$. By Bezout's theorem the intersection of S with these hypersurfaces consists of at most $n(n-1)^2$ points. Since these are singular on S, they are counted at least two times in the intersection, so the effective bound is $\frac{n}{2}(n-1)^2$. In the table below in the first row we give the number $\frac{n}{2}(n-1)^2$, $n = 6, 8, 12$, in the second row we give the length of the orbit of a point under G_n, which is not a fix point,

n	6	8	12
$\frac{n}{2}(n-1)^2$	75	196	726
orbit	144	576	3600

Clearly such a point cannot be singular.
Let now x denote a singular point and consider it as vector in $\mathbb{C}^4$. Let $\sigma := \sigma(p, q) \in G_n$ s.t. $\sigma x = \lambda x$ equivalently

$$pxq^{-1} = \lambda x.$$

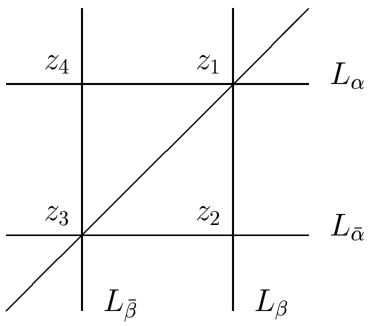
Consider x as matrix and take the determinant on booth sides of this equation. We get $\det(x) = \lambda^2 \det(x)$. In fact $\det(p) = \det(q) = 1$, since they are matrices in SU(2). The equality holds only when $\det(x) = 0$ or $\lambda^2 = 1$. If $\det(x) = 0$ then $x \in Q_n$ and this is not possible by lemma 4.1. □

We have seen in lemma 2.2 that if α is eigenvalue of a matrix $\sigma \in G_n \subseteq SO(4)$, then $\bar{\alpha}$ is eigenvalue too. Hence if $\alpha = 1$, resp. -1, then $\bar{\alpha} = 1$, resp. -1 too.

If $\sigma \neq \pm\mathbb{I}$ the corresponding eigenspace in $\mathbb{C}^4$ has dimension two (cf. lemma 2.2), so is a line in $\mathbb{P}^3_{\mathbb{C}}$. This shows that the singular points are contained in lines of fix points of elements of G_n. In the next section we describe these lines.

4.1 Lines of fix points

We describe first how the fix lines of the elements $\sigma(p,q)\neq\pm\mathbb{I}$ of G_n, which correspond to eigenspaces in $\mathbb{C}^4$ with eigenvalues 1 or -1, are determined by the fix lines of the elements $\sigma(p,\mathbb{I})$ and $\sigma(\mathbb{I},q)$. The element $\sigma(p,\mathbb{I})$ has two fix lines in the ruling $\{v\} \times \mathbb{P}^1$ of Q_2 and the element $\sigma(\mathbb{I},q)$ has two fix lines in the ruling $\mathbb{P}^1 \times \{w\}$. Let α, $\bar{\alpha}$ be the eigenvalues of $\sigma(p,\mathbb{I})$ and L_α, $L_{\bar{\alpha}}$ the corresponding eigenspaces in $\mathbb{C}^4$, let β, $\bar{\beta}$ be the eigenvalues of $\sigma(\mathbb{I},q)$ and L_β, $L_{\bar{\beta}}$ the corresponding eigenspaces in $\mathbb{C}^4$. Moreover let $z_1 \in L_\alpha \cap L_\beta$, $z_2 \in L_{\bar{\alpha}}\cap L_\beta$, $z_3 \in L_{\bar{\alpha}}\cap L_{\bar{\beta}}$, $z_4 \in L_\alpha\cap L_{\bar{\beta}}$. The $z_i's$ are eigenvectors of the matrix $\sigma(p,q)$ with eigenvalues $\alpha\beta$, $\bar{\alpha}\beta$, $\bar{\alpha}\bar{\beta}$ and $\alpha\bar{\beta}$. If one of these eigenvalues is 1 or -1, then the conjugate eigenvalue is 1 or -1 too, so $\sigma(p,q)$ has a whole line of fix points in $\mathbb{P}^3_{\mathbb{C}}$. E.g. if $\bar{\alpha}\bar{\beta}= \alpha\beta= 1$ or -1, then the line $< z_1, z_3 >$ is a fix line of $\sigma(p,q)$. See picture below



Using this construction we give the following lemma that we will use later. We use the same notation as above.

Lemma 4.2 *Let L denote a fix line of $\sigma(p,q)\in G_n$ and assume that L meets the base locus of the pencil $F_n(\lambda)$. W.l.o.g. let these intersection points be z_1, z_3 as above. Then for each surface $S \neq Q_n$ in the pencil we have $mult_{z_i}(L \cdot S) = 1$.*

Proof. By lemma 4.1 the points z_i, $i = 1, 3$, are smooth points on each surface (not Q_n) in the pencil $F_n(\lambda)$. The lines of the two rulings of Q_2 which meet at z_i, $i = 1, 3$, are lines of the base locus, hence are contained in S. The tangent space of S at the z_i is the plane spanned by these two lines. Clearly this plane does not contain L, hence L cannot be tangent to S at z_i, $i = 1, 3$. $\square$

In the table below we give the conjugacy classes under G_n of the elements in G_n, which have eigenspaces of dimension 2 in $\mathbb{C}^4$ with eigenvalue 1 or -1 (cf. table on page 15 and proposition 2.2). We use the notation of chapters 1 and 2. In particular when we write $[\pm\sigma]$ we mean the two distinct conjugacy classes $[+\sigma]$ and $[-\sigma]$. Of course these elements have the same fix lines. Observe that the elements in the conjugacy classes $[\pi_3\pi_3']$, $[\pi_3^2\pi_3'^2]$; $[\pi_5\pi_5']$, $[\pi_5^2\pi_5'^2]$; $[\pi_3\pi_3'^2]$, $[-\pi_3^2\pi_3']$, have two by two the same fix lines, so we consider them together. The elements in the conjugacy classes $[\sigma_{24}]$, $[\pi_4\pi_4']$ have the same fix lines too. The latter is not trivial and we prove it.
Proof. The element $\pi_4\pi_4'= \sigma(p_4, p_4)$ has just one line of fix points which is a line of fix points of $\sigma(p_4^2, p_4^2) = \sigma_1\sigma_3 \in [\sigma_{24}]$ too. The element $-\sigma(p_4, p_4^3)$ $\in [\sigma(p_4, p_4)]$ has the other line of fix points of $\sigma_1\sigma_3$ as fix line. We identify $p_4 \in$SU(2) with a permutation of S_4. In fact via the map ρ defined on page 6, $\rho(p_4):= R_4\in$SO(3) (cf. chapter 1) and it corresponds to the permutation (1234) of section 1.3. In S_4 the square of an element of order 4 is an even permutation of order 2 and each such permutation is the square of a permutation of order 4. Hence the fix lines of the elements in $[\pi_4\pi_4'] = [\sigma(p_4, p_4)]$ are the same as in $[\sigma_{24}]$. $\square$

Under the conjugacy classes we write the number of the distinct eigenspaces in $\mathbb{C}^4$, a short explanation of this follows.

	Conjugacy classes with eigenvalues ± 1		
G_6	$[\sigma_{24}]$	$[\pm\pi_3\pi_3']$,$[\pm\pi_3^2\pi_3'^2]$	$[\pm\pi_3^2\pi_3']$, $[\pm\pi_3\pi_3'^2]$
	18	16	16
G_8	$[\sigma_{24}]$,$[\pm\pi_4\pi_4']$	$[\pm\pi_3\pi_3']$	$[\pi_3\pi_4\pi_3'\pi_4']$
	18	32	72
	$[\pi_3\pi_4\sigma_4]$	$[\sigma_2\pi_3'\pi_4']$	
	36	36	
G_{12}	$[\sigma_{24}]$	$[\pm\pi_5^2\sigma_2\pi_5'^2\sigma_4]$	$[\pm\pi_5\pi_5']$, $[\pm\pi_5^2\pi_5'^2]$
	450	200	72

In the case of $[\sigma_{24}]$, $[\pi_3\pi_4\pi_3'\pi_4']$, $[\pi_3\pi_4\sigma_4]$, $[\sigma_2\pi_3'\pi_4']$ each matrix has two distinct fix lines. Since σ, $-\sigma$ are in the same conjugacy class, if n is the total number of elements in a conjugacy class, we get just $2 \cdot \frac{n}{2}= n$ distinct fix lines. The conjugacy classes $[\pi_3\pi_3']$ and $[\pi_3\pi_3'^2]$ under G_6 contain 16 elements each, with just one line of fix points (this follows from the table on page 15). So we have 16 distinct lines of fix points. Observe that if we use again the notation of the construction on page 39 and the fix line of $\pi_3\pi_3'$ passes through the points z_1, z_3, the fix line of $\pi_3\pi_3'^2$ passes through the points z_2, z_4. This shows, in particular, that these matrices have distinct fix lines.
The conjugacy classes $[\pi_3\pi_3']$ under G_8 and $[\pi_5^2\sigma_2\pi_5'^2\sigma_4]$, $[\pi_5\pi_5']$ under G_{12} have order respectively 64, 400 and 144. Now the elements $\pi_3^2\pi_3'^2$, $(\pi_5^2\sigma_2)^2(\pi_5'^2\sigma_4)^2$ and $\pi_5^4\pi_5'^4$ are in these conjugacy classes and have the same fix lines, hence we get just $\frac{64}{2} = 32$, $\frac{400}{2}= 200$ and $\frac{144}{2}= 72$ distinct fix lines for the elements in each conjugacy class.

Lemma 4.3 *If $[\sigma]$ denotes a conjugacy class in the previous table, then the fix lines of the elements in $[\sigma]$ form one orbit under G_n. Moreover, the fix lines of the elements in $[\sigma_2\pi_3'\pi_4']$ are in the orbit of the fix lines of the elements in $[\pi_3\pi_4\sigma_4]$ under the action of the matrix C of page 17.*

Proof. The statement is clear when σ has just one line of fix points. Moreover $C\pi_3\pi_4\sigma_4C^{-1}= \sigma_2\pi_3'\pi_4'$, so the last assertion is clear too. We are left to prove that the two fix lines of the elements in $[\sigma_{24}]$, $[\pi_3\pi_4\pi_3'\pi_4']$ or $[\pi_3\pi_4\sigma_4]$ are equivalent under G_n. The latter are eigenspaces of $\mathbb{C}^4$ with eigenvalues 1 and -1. Remember that if π is an element in one of the previous classes then $-\pi$ is in the same conjugacy class. It has the same eigenspaces but with eigenvalues interchanged. So we can find a matrix in G_n which maps one line to the other and vice versa. □

This in particular shows that every G_n-invariant property, which holds for a special fix line of an element in a conjugacy class above, holds for each other fix line of the elements in the same conjugacy class.

Lemma 4.4 *The intersection points of the previous lines are real. In particular they are not on the quadric.*

Proof. Let $L\neq L'$ be fix lines of the elements σ, $\sigma' \in G_n$. They descend from eigenspaces in $\mathbb{C}^4$ with eigenvalue 1 or -1. Let $x \in L \cap L'$. Considering x as eigenvector in $\mathbb{C}^4$, we have $\sigma x = \alpha x$ and $\sigma' x = \beta x$, with $\alpha,\beta= \pm 1$. We have also $\alpha\bar{x} = \bar{\sigma}\bar{x} = \sigma\bar{x}$ and $\beta\bar{x} = \bar{\sigma}'\bar{x} = \sigma'\bar{x}$, since the matrices σ and σ' are real. Hence $\bar{x}$ is eigenvector with the same eigenvalue as x. As point of $\mathbb{P}^3_{\mathbb{C}}$, this means that $\bar{x} \in L \cap L'$ too, but $L \neq L'$ therefore $x = \bar{x}$, so x is a real point. □

We will see that the singular points of the surfaces in the pencils are intersection points of these lines. This lemma shows that they are real points. From now on, if not explicitly stated, we consider just fix lines of elements belonging to the conjugacy classes in the table on page 40 (eigenvalues ± 1).

4.2 Configurations

We recall the definition of space configuration of lines and points. For more details cf. [7], p.12.

Definition 4.1 A *space configuration* of lines and points is a system of l lines and p points s.t. each line contains π of the given points and each point belongs to λ lines. We say that we have a (p_λ, l_π) configuration. In this situation

$$p \cdot \lambda = l \cdot \pi. \tag{7}$$

On pages 38, we have seen that the singular points are on lines of fix points. Assume that $\mathcal{S}$ is a G_n-orbit of singular points with $|\mathcal{S}| := N_0$. Let N_1 be the number of distinct lines fixed by some elements in a conjugacy class $[\sigma]$. They form one orbit under the action of the group G_n (cf. lemma 4.3), therefore if the line L, with $\sigma L = L$ contains n_0 of the N_0 points, each other fix line contains n_0 points too. Moreover we have

Lemma 4.5 *Let $p \in \mathcal{S}$ and assume that n_1 of the fix lines of the elements in $[\sigma]$ contain p. Then through each other point of $\mathcal{S}$ pass n_1 lines.*

Proof. Assume that $p \in L$ with $\sigma L = L$ and let $q \in \mathcal{S}$, $q \neq p$. There is a $\gamma \in G_n$ with $q = \gamma p$. Clearly $q \in \gamma L$, which is a fix line of the element $\sigma' = \gamma \sigma \gamma^{-1} \in [\sigma]$. Therefore if n_1 lines pass through p, then n_1 lines pass through $\gamma p = q$. □

In conclusion the fix lines and a set of singular points like $\mathcal{S}$ form a configuration of lines and points in sense of definition 4.1. Writing again formula (7), we have

$$N_1 \cdot n_0 = N_0 \cdot n_1. \tag{8}$$

If the n_0 singular points belong to an invariant surface in $F_n(\lambda)$, then all the N_0 points belong to the same surface, so we know that in the pencil we have a surface with at least N_0 singular points.

Cycles of A_4, S_4, A_5 and points of $\mathbb{P}^3_{\mathbb{C}}$

In chapter 1 we showed how a cycle in a permutation group A_4, S_4 or A_5 has a representation as a matrix of SO(3). By the map ρ defined on page 6 it corresponds to two matrices in SU(2) and just to one in $\mathbb{P}$GL(2). At the beginning of chapter 3 we identified this space with $\mathbb{P}^3_{\mathbb{C}}\backslash\{\mathbb{P}^1 \times \mathbb{P}^1\}$. In this way we can think of A_4, S_4, A_5 as subsets of $\mathbb{P}^3_{\mathbb{C}}$ and of a cycle as a point of $\mathbb{P}^3_{\mathbb{C}}$. We will use this fact several times in section 4.4. Here we show

Proposition 4.4 *The points of A_4, S_4 and A_5 form one orbit under the action of G_6, G_8 and G_{12}.*

Proof. For $\mathbb{I}\in A_4$, S_4, or A_5 and $q\in A_4$, S_4 or A_5 one has $\sigma(q,\mathbb{I})\mathbb{I}= q$, with $\sigma(q,\mathbb{I})\in G_6$, G_8, G_{12}. So the assertion follows. □

Proposition 4.5 *The points of $S_4\backslash A_4$ form one orbit under the action of G_6.*

Proof. The point p_3p_4 of $\mathbb{P}^3_{\mathbb{C}}$ corresponds to the cycle (34) of $S_4\backslash A_4$ and the matrices $\sigma(q,\mathbb{I})$, $q\in\tilde{A}_4\subseteq\mathbb{P}\mathrm{GL}(2)$ correspond to cycles of A_4. Observe that the product $\sigma(q,\mathbb{I})p_3p_4 = qp_3p_4$ is in $S_4\backslash A_4$, in fact considering the sign of the permutations we find $\mathrm{sgn}(q)\cdot\mathrm{sgn}((34)) = 1\cdot(-1) = -1$. Moreover for two different cycles $q, q' \in A_4$ we have $qp_3p_4 \neq q'p_3p_4$, therefore we can write:

$$S_4\backslash A_4 = \{\sigma(q,\mathbb{I})p_3p_4 | q \in A_4\}$$

This shows that the points of $S_4\backslash A_4$ form one orbit under the action of G_6. □

4.3 Coverings of $\mathbb{P}^1$

A point $x_0\in\mathbb{P}^3_{\mathbb{C}}$, not in the base locus of the pencil $F_n(\lambda)$, determines a surface $S_n(x) + \lambda Q_n(x) = 0$, $\lambda = -\frac{S_n(x_0)}{Q_n(x_0)}\in\mathbb{P}^1$, passing through x_0. If $x_0\in Q_n$ then $\lambda=\infty$. Consider now a line of fix points L as in section 4.1. First assume that L does not meet the base locus. We define a morphism

$$f : L \longrightarrow \mathbb{P}^1 \tag{9}$$

via $f(x) = -\frac{S_n(x)}{Q_n(x)}$ if $x \notin Q_n$, $f(x)=\infty$ if $x \in Q_n$. The fiber over a point $\lambda\in\mathbb{P}^1$ consists of the points of intersection of the surface $S_n(x)+\lambda Q_n(x) = 0$ with the line L. In general, i.e. away from the ramification locus (for the

definition of ramification locus cf. e.g. [13], p. 299), this is a $n:1$ cover of $\mathbb{P}^1$. In fact the line L intersects a general surface of the pencil in $n = \deg F_n(\lambda)$ points.
If L meets the base locus $Q_n \cap S_n$ of $F_n(\lambda)$, then the map f is not defined at the points of intersection p_1, p_2. By lemma 4.2 the line L meets each surface in $F_n(\lambda)$ with multiplicity one at these points. So f extends to a cover

$$\bar{f} : L \longrightarrow \mathbb{P}^1 \tag{10}$$

of degree $n-2$ having branch points of order $\frac{n}{2}-1$ at p_1, p_2. We calculate now the degree of the ramification locus of the maps f and $\bar{f}$. For this we recall Hurwitz's theorem (cf. [13], p. 301):

Theorem 4.1 (Hurwitz) *Let $f : X \longrightarrow Y$ be a finite morphism of curves, and let $n = \deg f$, then :*

$$2g(X) - 2 = n \cdot (2g(Y) - 2) + \deg R,$$

with R the ramification locus, $\deg R = \sum_{p\in X}(e_p - 1)$ and e_p the ramification index.

Proposition 4.6 *The degree of the ramification locus of the morphism f is $2n-2$ and of $\bar{f}$ it is $2n-6$ $(n = 6, 8, 12)$.*

Proof. Apply Hurwitz's theorem to f and $\bar{f}$. □

Put now $\alpha := \sum_{p\notin Q_n}(e_p - 1)$ and let $L \cap Q_n = \{p_1, p_2\}$. If L does not meet the base locus then we have $e_{p_i} = \frac{n}{2}$, if L meets the base locus then $e_{p_i} = \frac{n}{2} - 1$. Using now the degree of the ramification locus which we found above we get in the first case $\alpha = n$ and in the second case $\alpha = n-2$. This gives an upper bound for the number of singular points which eventually occur on L.

Proposition 4.7 *The singular points of the surfaces in the pencils are ramification points of the morphism* (9), *resp.* (10).

Proof. The ramification locus of f, resp. $\bar{f}$ are the points of L where the Jacobian matrix has not maximal rank. A calculation shows that this is the case at the singular points. □

In fact, except for the points on the complex quadric, we will see that also the converse is true, i.e. each ramification point is a singular point.

4.4 Singular points

We have seen on page 38 that the singular points of the surfaces in the pencils $F_n(\lambda)$ lie on lines of fix points. These are the fix lines of the elements in the conjugacy classes in the table on page 40. By lemma 4.3 we have to analyze just one line in each conjugacy class. In section 4.3 we defined morphisms from a fix line L to $\mathbb{P}^1$ and we remarked that the degree of the ramification locus gives an upper bound for the number of singular points on these lines. In this chapter we take a special line in each conjugacy class and we give the singular points of the surfaces in the pencils $F_n(\lambda)$ on it. We found the singular points and the value of λ for which the surface is singular in $F_n(\lambda)$ in the following way. Let $\{f = 0, g = 0\}$ be the equations of the line L and $\partial_i S_n(x) + \lambda \partial_i Q_n(x)$, $i = 0, 1, 2, 3$, denote the partial derivatives with respect to x_0, x_1, x_2, x_3. The solutions $(\lambda, (x_0 : x_1 : x_2 : x_3))$ in $\mathbb{P}^1 \times L$ of the system

$$\begin{cases} f &= 0 \\ g &= 0 \\ F_n(\lambda) : S_n(x) + \lambda Q_n(x) &= 0 \\ \partial_i S_n(x) + \lambda \partial_i Q_n(x) &= 0 \quad (i = 0, 1, 2, 3) \end{cases}$$

give the singular points and the singular surfaces in the pencils. We calculate them using MAPLE. We give these results in the tables in chapter 6. The last column of each table will be explained in detail thorough **i),ii),iii),iv)** below. So when we refer to the tables of chapter 6, we mean only the first five columns. Using the latter, we give a first result

Proposition 4.8 *In each pencil $F_n(\lambda)$, $n = 6, 8, 12$, there is at least one surface with singular points which correspond to cycles in the permutation groups A_4, S_4 and A_5. More precisely*

surface	*sing. points*
$F_6(-1)$	A_4
$F_6(-\frac{1}{4})$	$S_4 \backslash A_4$
$F_8(-1)$	S_4
$F_{12}(0)$	A_5

Proof. We use the identification given on page 43, and the propositions 4.4 and 4.5. The surfaces $F_6(-1)$, $F_8(-1)$, $F_{12}(0)$ contain the point $(1 : 0 : 0 :$

0)$\in\mathbb{P}^3_{\mathbb{C}}$ which corresponds to the identity. The surface $F_6(-\frac{1}{4})$ contains the point $(-1:0:1:0)\in\mathbb{P}^3_{\mathbb{C}}$ which corresponds to the matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ i & i \end{pmatrix} \in \mathbb{P}\mathrm{GL}(2).$$

Via the map ρ we find the matrix of SO(3)

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

which corresponds, with notation of chapter 1, section 1.3, to a rotation of order 2 around an e-axis of the octahedron. □

Then we proceed as follows.
i) We show that the singular points of intersection of a special fix line with a singular surface form one orbit under the fix group of the line. Hence the singular points of this surface on the fix lines in the orbit of the given line form a G_n-orbit (cf. lemma 4.3)
ii) All the singular points on the lines in the tables $1, 2, 3$ of chapter 6 are double points.
iii) We explain how the fix lines meet each other, so that using the formula (8): $N_1 \cdot n_0 = N_0 \cdot n_1$, we find the singular points on a surface.
iv) We show, if necessary, that the fix lines of the elements in any other conjugacy class meet a singular surface in points of the G_n-orbit of **i)**.
Hence from the tables on chapter 6 and **ii)** using the degree of the ramification locus of the morphisms f, resp. $\bar{f}$ on these lines (cf. proposition 4.6) we get

- the ramification points, not on the multiple quadric, are double points on surfaces in the pencils. Their number is n if the line does not intersect the base locus, resp. $n-2$ if the line meets the base locus,
- we have no other ramification points on the fix lines, so no other singular points and singular surfaces in the pencils.

This, together with the tables in chapter 6, shows that:

in each pencil there are exactly four singular surfaces, with double points, which by **i)** *and* **iv)** *form one orbit under* G_n.

To show **i)-iv)** the following facts will be useful. We use the notation of chapter 1.

1) Consider a fix line $L_{\pi\pi'}$ of an element $\sigma(\pi,\pi')$. Then for $x \in L_{\pi\pi'}$ the orbit of x under $\sigma(\pi,\mathbb{I})$, $\sigma(\mathbb{I},\pi')$ is again on $L_{\pi\pi'}$. In fact these last matrices commute with $\sigma(\pi,\pi')$, hence leave the line $L_{\pi\pi'}$ invariant.
2) We have seen in section 4.1 that a matrix of order two in $\tilde{S}_4 \subseteq \mathbb{P}\mathrm{GL}(2)$ is the square of a matrix of order four. Consider the matrix $q_2 \in \tilde{S}_4$. There is a matrix $q \in \tilde{S}_4$ s.t. $q^2 = q_2$. The matrices $\sigma_2 := \sigma(q_2,\mathbb{I})$ and $\sigma(q,\mathbb{I})$, resp. $\sigma_4 := \sigma(\mathbb{I},q_2)$ and $\sigma(\mathbb{I},q)$ commute. As in **1)**, we can consider the orbit of a point on a fix line of the matrix σ_{24} under the action of $\sigma(q,\mathbb{I})$ and $\sigma(\mathbb{I},q)$, and of the matrices $\pi_3\pi_4\sigma_4$, resp. $\sigma_2\pi_3'\pi_4'$ under the action of $\sigma(\mathbb{I},q)$, resp. $\sigma(q,\mathbb{I})$.
3) Consider the matrices $\sigma(\pi,\pi')$, resp. $\sigma(\gamma,\gamma')$ in the same conjugacy class in the table on page 40 and the intersection point, x, of their fix lines $L_{\pi\pi'} \neq L_{\gamma\gamma'}$. We have $\sigma(\pi,\pi')x = x$ and $\sigma(\gamma,\gamma')x = x$ or equivalently

$$\pi x \pi'^{-1} = x \quad \text{and} \quad \gamma x \gamma'^{-1} = x. \tag{11}$$

Consider x as a 2×2-complex matrix. Since $x \notin Q_n$ (cf. lemma 4.1), it is an invertible matrix, hence we can write the (11) as

$$x^{-1}\pi x = \pi' \quad \text{and} \quad x^{-1}\gamma x = \gamma'.$$

Now consider the groups $< \pi, \gamma >:= G_1$ and $< \pi', \gamma' >:= G_2$. The point x defines an inner automorphism of SU(2)

$$\begin{array}{rccc} \mathrm{int}_x : & \mathrm{SU}(2) & \longrightarrow & \mathrm{SU}(2) \\ & q & \longmapsto & x^{-1}qx \end{array}$$

which restricts to G_1 as

$$\mathrm{int}_x|G_1 : \quad G_1 \longrightarrow G_2. \tag{12}$$

We identify the elements π, γ, π', $\gamma' \in \tilde{G} \subseteq \mathrm{SU}(2)$ with permutations of A_4, S_4 or A_5. So the morphism (12) is in fact an isomorphism between subgroups of these permutation groups. If $G_1 = G_2 = A_4, S_4$ or A_5, by [24], 11.4.6, p. 313, the automorphism is an inner automorphism given by an element $y \in A_4$, S_4 or A_5. Consider y as matrix of $\mathbb{P}\mathrm{GL}(2)$. We have

$$y^{-1}\pi y = \pi' \quad \text{and} \quad y^{-1}\gamma y = \gamma',$$

This means that y as point of $\mathbb{P}^3_{\mathbb{C}}$ belongs to $L_{\pi\pi'} \cap L_{\gamma\gamma'}$. Since the lines are different, necessarily $x = y$. So $x \in A_4$, S_4 or A_5. The automorphism (12) tells us in fact much more: for each element $p \in G_1$ we have $x^{-1}px = q \in G_2$, so x belongs to the fix line of $\sigma(p,q)$ too.

4) In section 4.1 we showed that the fix lines of the elements in $[\sigma_{24}]$ are the same as those of the elements in $[\pi_4\pi_4']$. The fix lines of the matrices in $[\sigma_2\pi_3'\pi_4']$ are in the orbit of the fix lines of the matrices in $[\pi_3\pi_4\sigma_4]$ under the matrix C (cf. lemma 4.3). Moreover the surfaces of the pencils are invariant under C.
We will use these facts several times in the sequel. However, to understand **i)-iv)** correctly, one has to keep in mind the tables of chapter 6.

The situation is quite easy for the pencil $F_6(\lambda)$. We can apply **1)** above to each fix line and see that the points of intersection with a singular surface form one orbit (cf. table 6.1 on pages 65-65). This shows **i)**. Moreover since in this way they have all the same multiplicity, using Bezout's theorem and lemma 4.2 (if the line meets the base locus), we see that the singular points are all double points, this proves **ii)**. This, together with proposition 4.8, completes the case of the surfaces $F_6(-1)$ and $F_6(-\frac{1}{4})$: they have 12 double points. Using **3)**, we explain **iii)** for the surfaces $F_6(-\frac{2}{3})$ and $F_6(-\frac{7}{12})$.

Proposition 4.9 *The fix lines of the elements in* $[\pi_3\pi_3']$ *meet at the points of* A_4*.*

Proof. Assume that x is an intersection point of $L_{\pi\pi'}\neq L_{\gamma\gamma'}$, the fix lines of the elements $\sigma(\pi,\pi')$, $\sigma(\gamma,\gamma')\in[\pi_3\pi_3']$. Then

$$x^{-1}\pi x=\pi' \quad \text{and} \quad x^{-1}\gamma x=\gamma'.$$

If $<\pi>=<\gamma>$ then $\pi=\gamma$ because they are in the same conjugacy class. So we get $\pi'=\gamma'$ too, but this is not possible since the fix lines are distinct. We assume $<\pi>\neq<\gamma>$ and so $<\pi'>\neq<\gamma'>$. If for some product $\pi^{a_1}\gamma^{a_2}=\pi^{b_1}\gamma^{b_2}$ then $\pi^{a_1-b_1}=\gamma^{b_2-a_2}$. This is possible if and only if $a_1=b_1$ and $a_2=b_2$. It follows that $|<\pi,\gamma>|\geq 9$, but $<\pi,\gamma>\subseteq A_4$ and A_4 contains no proper subgroups of order ≥ 9, so $<\pi,\gamma>=A_4$. In the same way we have $<\pi',\gamma'>=A_4$. By **3)**, the point x determines an automorphism

$$\begin{array}{ccc} A_4 & \longrightarrow & A_4 \\ q & \longmapsto & x^{-1}qx, \end{array}$$

with $x\in S_4$. Now, in terms of cycles of A_4, x does not interchange the conjugacy classes of (123) and (132) in A_4, so $x\in A_4$.

Proposition 4.10 *The fix lines of the elements in* $[\pi_3\pi_3'^2]$ *meet at the points of* $S_4\backslash A_4$*.*

Proof. The proof is the same as the proof of proposition 4.9 with the only difference that such an x interchanges the conjugacy classes, so $x \in S_4 \backslash A_4$. □

Using the formula (8) on page 42 with $N_1 = 16$, $n_0 = 3$ and $n_1 = 1$ we get $N_0 = 48$, the number of singular double points on $F_6(-\frac{2}{3})$ and $F_6(-\frac{7}{12})$.

The situation is more complicate for the pencils $F_8(\lambda)$ and $F_{12}(\lambda)$, $\lambda \in \mathbb{P}^1$. First, we show that the singular points on $F_8(-1)$, resp. on $F_{12}(0)$ are double points.

Proof. Consider the intersection with the fix lines of σ_{24}, resp. of $\pi_5\pi_5'$, then use Bezout's theorem and lemma 4.2. □

Combined with proposition 4.8, this completes the description of these surfaces.

The surface $F_8(-\frac{3}{4})$. By **1)** the points of intersection with the fix line of $\pi_4\pi_4'$ form one orbit and have the same multiplicity, hence this must be 2. This shows **i)** and **ii)**. About **iii)**, we have

Proposition 4.11 *The lines of fix points of the elements in $[\pi_4\pi_4']$ meet at the points of S_4.*

Proof. Let $L_{\pi_4\pi_4'} \neq L_{\gamma\gamma'}$ be the fix lines of $\pi_4\pi_4'$ and $\sigma(\gamma,\gamma') \in [\pi_4\pi_4']$. If $x = L_{\gamma\gamma'} \cap L_{\pi_4\pi_4'}$, then $x^{-1}\gamma x = \gamma'$ and $x^{-1}p_4x = p_4$. If $<\gamma> = <p_4>$ then $\gamma = p_4^m$, $m \in \mathbb{N}$, so $\gamma' = x^{-1}p_4^m x = (x^{-1}p_4x)^m = p_4^m$ and $\gamma = \gamma' = p_4^m$. Hence $L_{\gamma\gamma'} = L_{\pi_4\pi_4'}$, which contradicts the assumption. So $<\gamma> \neq <p_4>$ and $<\gamma'> \neq <p_4>$. Moreover if $\gamma^{a_1}p_4^{a_2} = \gamma^{b_1}p_4^{b_2}$ then $a_1 = b_1$ and $a_2 = b_2$. So we get $|<\gamma, p_4>| \geq 16$, hence $<\gamma, p_4> = S_4$. Similarly $<\gamma', p_4> = S_4$. By **3)** we have an automorphism

$$\begin{array}{ccc} S_4 & \longrightarrow & S_4 \\ q & \longmapsto & x^{-1}qx \end{array}$$

with $x \in S_4$. □

Using the formula (8), we find 72 double points on $F_8(-\frac{3}{4})$. In this case we have to show **iv)** too. By **4)**, we only need to show that the points of intersection of the fix lines of $\pi_3\pi_4\sigma_4$ are in the orbit of these 72 points under G_8.

Proof. Using **2)** we see that the points on these lines form one orbit. The fix lines of the element σ_{23} are $<(1:0:0:1),(0:1:1:0)>$ and

$< (0:-1:1:0), (-1:0:0:1) >$, these are fix lines of a matrix in $[\pi_4\pi_4']$ (cf. **4)**).They meet the fix lines of $\pi_3\pi_4\sigma_4$ at the points $(1:a:a:1)$, resp. $(a:1:-1:-a)$, $a := 1+\sqrt{2}$. □

The surface $F_8(-\frac{5}{9})$. To prove **i)** and **ii)** we apply **1)** and lemma 4.2 to the singular points on the fix line of $\pi_3\pi_3'$. They form one orbit and hence have multiplicity 2. By proposition 4.9 and proposition 4.10 these fix lines meet in points of S_4, so we find 96 singular points on $F_8(-\frac{5}{9})$ (use the formula (8)), this gives **iii)**. To prove **iv)**, we show that the points of intersection of the fix lines of $\pi_3\pi_4\pi_3'\pi_4'$ with $F_8(-\frac{3}{4})$ are in this orbit.
Proof. Observe that these 96 singular points are the 48 of the surfaces $F_6(-1)$ and $F_6(-\frac{1}{4})$ together. By table 1, we know that the point $(0:1:2:1) \in F_8(-\frac{5}{9})$ too and is one of the 96. Using the notation of chapter 1 we put $\sigma_{134} := \sigma_1\sigma_3\sigma_4$, $\sigma_{123} := \sigma_1\sigma_2\sigma_3$. Then

$$\begin{array}{lcllcl} \sigma_{134}(0:1:-1:1) & = & (-1:1:0:1), & \sigma_{123}(0:1:-1:1) & = & (1:1:0:1), \\ \sigma_{134}(0:1:2:1) & = & (-2:1:0:1), & \sigma_{123}(0:1:2:1) & = & (2:1:0:1), \end{array}$$

which shows that the four points of intersection of the fix lines of $\pi_3\pi_4\pi_3'\pi_4'$ with $F_8(-\frac{5}{9})$ are in the orbit of the 96 above. □

The surface $F_8(-\frac{9}{16})$. By **2)** the points of intersection of the fix lines of $\pi_3\pi_4\sigma_4$ with $F_8(-\frac{9}{16})$ form one orbit. So since they have all the same multiplicity this must be 2, we get **i)** and **ii)**. We prove **iv)**. By **4)**, we have only to show that the singular points on the fix lines of $[\pi_3\pi_4\pi_3'\pi_4']$ are in the orbit under G_8 of the previous points.
Proof. The lines of fix points of $\sigma_2\pi_3\pi_4\pi_3'\pi_4' \in [\pi_3\pi_4\pi_3'\pi_4']$ are $< (0:0:0:1), (-1:0:1:0) >$ and $< (1:0:1:0), (0:1:0:0) >$ and they contain the points $(1:0:-1:-\sqrt{2})$ resp. $(1:\sqrt{2}:1:0)$ of a line of fix points of $\pi_3\pi_4\sigma_4$. The fix lines of $\sigma_2\pi_3\pi_4\pi_3'\pi_4'$ meet the surface $F_8(-\frac{9}{16})$ in two points which by **1)** form one orbit, so the assertion follows. □

To get **iii)** we have,

Proposition 4.12 *The lines of fix points of the elements in $[\pi_3\pi_4\pi_3'\pi_4']$ do not meet at the points of $F_8(-\frac{9}{16})$.*

Proof. We use **3)**. Let $x = L_{\pi_3\pi_4\pi_3'\pi_4'} \cap L_{pq}$, with L_{pq} a fix line of $\sigma(p,q) \in [\pi_3\pi_4\pi_3'\pi_4']$, and $L_{\pi_3\pi_4\pi_3'\pi_4'}$ a fix line of $\pi_3\pi_4\pi_3'\pi_4'$. We have $x^{-1}p_3p_4x = p_3p_4$ and $x^{-1}px = q$. Consider the group $< p_3p_4, p >$ and $< p_3p_4, q >$, where $p, q \in [p_3p_4]$. Assume $< p_3p_4 > \neq < p >$ and $< p_3p_4 > \neq < q >$, so $| < p_3p_4, p > | \geq 4$, $| < p_3p_4, q > | \geq 4$. We now identify the matrices of

$\tilde{S}_4$ with cycles of S_4. The element p_3p_4 corresponds to the cycle (34) and we have two possibilities: $p = (12)$, then $| < p_3p_4, q > | = 4$; $p = (13)$,(23),(14) or (24), then $< p_3p_4, p > \cong S_3$ ($| < p_3p_4, p > | = 6$).
a) Let $p = (12)$, then $< p_3p_4, p > = < (34), (12) > = \{(12); (34); (12)(34); \mathbb{I}\}$ $\cong D_2$ (the dihedral group with $2 \cdot 2 = 4$ elements), so necessarily $q = (12)$ too. Since $x^{-1}p_3p_4x = p_3p_4$ and $x^{-1}px = p$ we get $x^{-1}p_3p_4px = p_3p_4p$ and in terms of cycles $x^{-1}(12)(34)x = (12)(34)$. This means that x belongs to a fix line of σ_{24}. These fix lines contain points of $F_8(-1)$ or $F_8(-\frac{3}{4})$ and no other singular points. So $x \notin F_8(-\frac{9}{16})$.
b) Let $S := < p_3p_4, p > \cong S_3$. We have $S' := < p_3p_4, q > \cong S_3$ too. Now x defines an automorphism:

$$\begin{array}{ccc} S & \longrightarrow & S' \\ q & \longmapsto & x^{-1}qx \end{array}$$

and x belongs to a fix line $L_{\tau_1\tau_2}$ of $\sigma(\tau_1, \tau_2) \in [\pi_3\pi_3']$. Again these lines contains singular points of $F_8(-1)$ and $F_8(-\frac{5}{9})$ and no other singular points. So $x \notin F_8(-\frac{9}{16})$. □

There are 72 fix lines of the elements in $[\pi_3\pi_4\pi_3'\pi_4']$. So we have a $(72_2, N_{01})$ configuration with the singular points on $F_8(-\frac{9}{16})$ found above. By the formula $72 \cdot 2 = N_0 \cdot 1$, we find $N_0 = 144$.

<u>The surface $F_{12}(-\frac{2}{25})$.</u> The points on the fix line of $\pi_5\pi_5'$ form one orbit by **1)** , so they have all the same multiplicity which must be 2. This shows **i)** and **ii)**. About **iii)** we have

Proposition 4.13 *The lines of fix points of the elements in $[\pi_5\pi_5']$ meet at the points of A_5.*

Proof. Let $L_{\pi_5\pi_5'} \neq L_{\tau_1\tau_2}$ denote the fix lines of $\pi_5\pi_5' := \sigma(p_5, p_5)$ and of the element $\sigma(\tau_1, \tau_2) \in [\pi_5\pi_5']$. Let $x = L_{\pi_5\pi_5'} \cap L_{\tau_1\tau_2}$. We can assume as usual $< p_5 > \neq < \tau_1 >$ and $< p_5 > \neq < \tau_2 >$, moreover $p_5^{a_1}\tau_1^{a_2} = p_5^{b_1}\tau_2^{b_2}$ holds if and only if $a_1 = b_1$ and $a_2 = b_2$, therefore we have $| < p_5, \tau_i > | \geq 25$. Here $| < p_5, \tau_i > | = 30$ is impossible, because A_5 does not contain subgroups of order 30. Hence we have $< p_5, \tau_i > = A_5$ ($i = 1, 2$). By **3)**, the point x defines an automorphism

$$\begin{array}{ccc} A_5 & \longrightarrow & A_5 \\ q & \longmapsto & x^{-1}qx. \end{array}$$

This automorphism is conjugation by some element of S_5. However, it does not interchange the conjugacy classes in A_5. So x, in fact, belongs to $A_5 \subseteq \mathbb{P}^3_{\mathbb{C}}$.
□

Using this proposition and the formula (8), we find 360 singular points on $F_{12}(-\frac{2}{25})$. We show that the points of intersection of the fix lines of the elements in $[\sigma_{24}]$ and $F_{12}(-\frac{2}{25})$ are some of these 360. This shows **iv)**.
Proof. Observe that the point $x = (0:0:\tau-1:1) \in L_{\pi_5\pi'_5} \cap F_{12}(-\frac{2}{25})$ is on the fix line $<(0:0:1:0),(0:0:0:1)>$ of $\sigma_{13} := \sigma_1\sigma_3$. Moreover the points $\sigma(q_1,\mathbb{I})x = (0:0:1:1-\tau)$, $\sigma_{124}x = \sigma_1\sigma_2\sigma_4 x = (0:0:1:\tau-1)$ and $\sigma_{124}(0:0:1:1-\tau) = (0:0:1-\tau:1)$ are on the fix line of σ_{13} too. This proves the assertion. □

The last two surfaces in $F_{12}(\lambda)$ are more difficult to analyze.

The surface $F_{12}(-\frac{3}{32})$. The points of intersection with the fix line $L_{\pi_3\pi'^2_3}$ of $\pi_3\pi'^2_3 \in [\pi_5^2\sigma_2\pi'^2_5\sigma_4]$ are

$$\begin{array}{lllllll} Q_0 & := & (0:1:1:0), & & Q_1 & := & (0:-1:0:1), \\ Q_2 & := & (0:0:1:1), & & Q_3 & := & (0:\tau+1:2+\tau:1), \\ Q_4 & := & (0:\tau-3:\tau-2:1), & & Q_5 & := & (0:\frac{1}{5}(\tau-3):\frac{1}{5}(2+\tau):1). \end{array}$$

The first three and the last three points form one orbit under the action of $\sigma(p_3,\mathbb{I})$. The matrix

$$\sigma(c,c) = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1-\tau & -\tau & 1 \\ 0 & -\tau & 1 & \tau-1 \\ 0 & 1 & \tau-1 & \tau \end{pmatrix} \in G_{12},$$

with $c = q_1 p_5^{-1} \in \tilde{A}_5$ is s.t. $\sigma(c,c)Q_0 = Q_4$. So the points Q_0, Q_1, Q_2, Q_3, Q_4, Q_5 form one orbit on $L_{\pi_3\pi'^2_3}$. In particular they have all the same multiplicity, so this must be 2. This shows **i)** and **ii)**. We show that these lines meet in such a way that the surface $F_{12}(-\frac{3}{32})$ has at least 300 singular points. This explains **iii)**.
Proof. A calculation with MAPLE shows that the points Q_i are in fact nodes on $F_{12}(-\frac{3}{32})$ (a more precise explanation will be given at the end of this chapter on pages 54-55). Since $\deg F_{12}(-\frac{3}{32}) = 12$ they are at most 645 by Miyaoka's bound (cf. [21]). Hence some of the lines of fix points of the elements in $[\pi_5^2\sigma_2\pi'^2_5\sigma_4]$ must intersect at these points. Let $x \in L_{\tau_1\tau_2} \cap L_{\pi_5^2\sigma_2\pi'^2_5\sigma_4}$ where $L_{\tau_1\tau_2}$ denotes the fix line of $\sigma(\tau_1,\tau_2) \in [\pi_5^2\sigma_2\pi'^2_5\sigma_4]$. Then $x^{-1}p_5^2q_2x = p_5^2q_2$, and $x^{-1}\tau_1 x = \tau_2$. We assume $<p_5^2q_2> \neq <\tau_1>$ and $<p_5^2q_2> \neq <\tau_2>$. We have the following possibilities:
a) $<p_5^2q_2,\tau_1> = A_5$ therefore $<p_5^2q_2,\tau_2> = A_5$ too. By **3)** we get $x \in A_5$, but this is not the case.

b) $G{:=}< p_5^2 q_2, \tau_1 > \cong A_4$ and $G'{:=}< p_5^2 q_2, \tau_2 > \cong A_4$ too. The point x describes an isomorphism

$$\begin{array}{ccc} G & \longrightarrow & G' \\ q & \longmapsto & x^{-1}qx. \end{array}$$

In particular x belongs to four lines of fix points of elements in the conjugacy class $[\pi_5^2\sigma_2\pi_5'^2\sigma_4]$ and to three lines of fix points of elements in the conjugacy class $[\sigma_{24}]$. Observe that we cannot have more lines of fix points containing x. Otherwise applying again **3)**, we find $x \in A_5$. Using formula (8) we have $200 \cdot 6 = N_0 \cdot 4$, so $N_0 = 300$. □

To prove **iv)**, we show that the two points of intersection of the fix lines of the elements in $[\sigma_{24}]$ with $F_{12}(-\frac{3}{32})$ are in the orbit of the 300.
Proof. We have seen that a singular point of the 300 is contained in three fix lines for elements in $[\sigma_{24}]$. These lines, by table 3, contain two singular points of $F_{12}(-\frac{3}{32})$, which, by **1)**, form one orbit. This proves the assertion. □

<u>The surface $F_{12}(-\frac{22}{243})$.</u> Consider the points of intersection of $F_{12}(-\frac{22}{243})$ and the fix line of $\pi_5^2\sigma_2\pi_5'^2\sigma_4$. By **1)** they form an orbit of length three under $\pi_5^2\sigma_2$. So we get **i)**.
Proof of **ii)**. Consider the morphism f of page 43 on this line. The degree of the ramification locus is 12 (not considering the intersection with Q_{12}). The points of intersection with the surfaces $F_{12}(0)$ and $F_{12}(-\frac{3}{32})$ have multiplicity 2, hence an easy calculation shows that the points on $F_{12}(-\frac{22}{243})$ have multiplicity 2 too. □

Now we want to explain **iii)**.

Proposition 4.14 *The lines of fix points in $[\pi_5^2\sigma_2\pi_5'^2\sigma_4]$ do not meet at the points of $F_{12}(-\frac{22}{243})$.*

Proof. The proof is very similar to the proof in the description of the surface $F_{12}(-\frac{3}{32})$ on page 52. So, with the same notation, we have to show that in case **b)**, the point x is not on $F_{12}(-\frac{22}{243})$. On the contrary assume that x is one of the points in $F_{12}(-\frac{22}{243}) \cap L_{\pi_5^2\sigma_2\pi_5'^2\sigma_4}$. Then we have a $(200_3, N_{04})$ configuration with the fix lines of the elements in $[\pi_5^2\sigma_2\pi_5'^2\sigma_4]$ and using formula (8) on page 42 we find $N_0 = 150$. With the fix lines of the elements in $[\sigma_{24}]$, we have a $(450_{n_0}, 150_3)$ configuration and so $n_0 = 1$. But if a line of fix points of an element $\sigma(q, q') \in [\sigma_{24}]$ contains the point x then it contains the point $\sigma(q, \mathbb{I})x$ too, therefore $n_0 \geq 2$ and we get a contradiction. So $x \notin F_{12}(-\frac{22}{243})$ and $N_0 = 600$. □

This shows that we have at least 600 singular points on $F_{12}(-\frac{22}{243})$. Finally we prove **iv)**. We show that the points on the fix lines of the elements in $[\sigma_{24}]$ are in the orbit of these 600 points.

Proof. The point $x = (0:1:\tau-2:0) \in L_{\pi_5^2\sigma_2\pi_5'^2\sigma_4} \cap F_{12}(-\frac{22}{243})$ is contained in the fix line $< (0:1:0:0), (0:0:1:0) >$ of $\sigma_{1234} := \sigma_1\sigma_2\sigma_3\sigma_4 \in [\sigma_{24}]$. Following notation of chapter 1, put $\sigma_{12} := \sigma_1\sigma_2$ and $\sigma_{14} := \sigma_1\sigma_4$. The points $\sigma_{12}x = (0:2-\tau:1:0)$, $\sigma_{14}x = (0:\tau-2:1:0)$ and $\sigma_{14}(0:2-\tau:1:0) = (0:1:2-\tau:0)$ are on this line too. □

We now show that the singular points are in fact all *nodes* (= *ordinary double points* A_1).

Definition 4.2 A point p on a surface $S \subseteq \mathbb{P}^3_{\mathbb{C}}$ given by the equation $\{F = 0\}$ is a *node* if it is a singular point and in an affine neighborhood of p with coordinate x, y, z, the rank of the Hesse matrix H at p is maximal, more precisely:

$$\operatorname{rank} H_{|p} = \operatorname{rank} \begin{pmatrix} \frac{\partial^2 F}{\partial x \partial x} & \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x \partial z} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y \partial y} & \frac{\partial^2 F}{\partial y \partial z} \\ \frac{\partial^2 F}{\partial z \partial x} & \frac{\partial^2 F}{\partial z \partial y} & \frac{\partial^2 F}{\partial z \partial z} \end{pmatrix}_{|p} = 3$$

Equivalently p is a node if it is analytically isomorphic to the vertex of a quadratic cone (cf. [1], lemma 3′).

We show that the point $\mathbb{I} = (1:0:0:0)$ is a node. In affine coordinates $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$, $z = \frac{x_3}{x_0}$ is the origin $(0,0,0)$. The equation of the surface $F_6(-1)$ becomes

$$\begin{aligned} 0 &= 1 + x^6 + y^6 + z^6 + 15(x^2y^2 + x^2z^2 + y^2z^2 + x^2y^2z^2) \\ &\quad -(1 + x^2 + y^2 + z^2)^3 \\ &= -3(x^2 + y^2 + z^2) + \text{terms of degree} \geq 3. \end{aligned}$$

The Hesse matrix at the origin is

$$\begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix},$$

which clearly has rank 3.

The surface $F_8(-1)$ has equation

$$\begin{aligned} 0 &= 1 + x^8 + y^8 + z^8 + 14(x^4 + y^4 + z^4 + x^4y^4 + x^4z^4 + y^4z^4) \\ &\quad +168x^2y^2z^2 - (1 + x^2 + y^2 + z^2)^4 \\ &= -4(x^2 + y^2 + z^2) + \text{terms of degree} \geq 3. \end{aligned}$$

The Hesse matrix at the origin has again rank 3, therefore the origin is a node. In the case of $F_{12}(0)$ the calculations are more complicate. Using the expression on page 31, we find the equation of $S_{12}= F_{12}(0)$ in the previous affine coordinates

$$0 \;=\; 2(x^2+y^2+z^2) + \text{terms of degree} \geq 3.$$

The Hesse matrix computed at the origin has rank 3 as in the previous cases, hence we have a node again. This shows that all the points of A_4, S_4 and A_5 are nodes on the surfaces $F_6(-1)$, $F_8(-1)$ and $F_{12}(0)$. For the others singular points in the pencils, the method is the same, we eventually translate the surface in such a way that the singular point coincides with the origin of an affine chart. In any case calculations with MAPLE show that the singularities of the surfaces in the pencils are nodes (in fact one has to check it just for one singularity on each surface, the other singularities are in its orbit under the action of G_n).

In the following table we collect the values of λ for which we have singular surfaces in the pencils $F_n(\lambda)$. We denote by $N_n(\lambda)$ the number of nodes on such a surface. Denote by S the sum of the numbers of singularities and by A the alternating sum of the numbers of singularities of the four singular surfaces in each pencil. For convenience we give these number here, in the last two columns of the table. We shall use them in the next chapter.

n							
6	λ	-1	$-2/3$	$-7/12$	$-1/4$	S	120
	$N_6(\lambda)$	12	48	48	12	A	0
8	λ	-1	$-3/4$	$-9/16$	$-5/9$	S	336
	$N_8(\lambda)$	24	72	144	96	A	0
12	λ	$-3/32$	$-22/243$	$-2/25$	0	S	1320
	$N_{12}(\lambda)$	300	600	360	60	A	0

5 Final Remarks

In this last section we consider the numbers S and A given in the table on page 55. We show that these are determined by the topology.

5.1 The number of nodes in the pencils $F_n(\lambda)$

Consider the variety

$$\Gamma_n := \{((\lambda:\mu),x) \in \mathbb{P}^1 \times \mathbb{P}^3 \mid \lambda Q_n(x) + \mu S_n(x) = 0\}.$$

We have projections

$$\begin{array}{ccccc} \Gamma_n & \subseteq & \mathbb{P}^1 & \times & \mathbb{P}^3 \\ & \mathrm{pr}_1 & \swarrow & & \searrow\ \mathrm{pr}_2 \\ & \mathbb{P}^1 & & & \mathbb{P}^3. \end{array}$$

The projection pr_1 is a flat morphism (for the definition of flat morphism cf. e.g. [13], p. 256) with fibres of dimension 2. Let $\Gamma_{n\lambda}$ denote the fibre $\mathrm{pr}_1^{-1}(\lambda)$ over a $\lambda \in \mathbb{P}^1$. By the theorem on fibre dimension (cf. [13], p.256, proposition 9.5) we get

$$\dim \Gamma_n \;=\; \dim \Gamma_{n\lambda} + \dim \mathbb{P}^1 \;=\; 3$$

On the other hand the fibre over a point $x_0 \in \mathbb{P}^3$, not in the base locus, is a point of $\mathbb{P}^1$. More precisely is the value $(\lambda:\mu) \in \mathbb{P}^1$ s.t. $\lambda Q_n(x_0)+\mu S_n(x_0) = 0$. So $\dim(\mathrm{pr}_2^{-1}(x)) = 0$, for all $x \in \mathbb{P}^3 \backslash \mathcal{B}_n$. If $x_0 \in \mathcal{B}_n$, all the surfaces of the pencil $F_n(\lambda)$ contain the point x_0, so $\mathrm{pr}_2^{-1}(x_0) \cong \mathbb{P}^1$. We can write $\Gamma_n = \mathbb{P}^1 \times \mathcal{B}_n \cup \Gamma_n \backslash \{\mathbb{P}^1 \times \mathcal{B}_n\}$. The restriction $\mathrm{pr}_2 : \Gamma_n \backslash \{\mathbb{P}^1 \times \mathcal{B}_n\} \longrightarrow \mathbb{P}^3 \backslash \mathcal{B}_n$ is a surjective morphism with fibres of dimension 0, therefore is an isomorphism. We now calculate $e := e(\Gamma_n)$, the topological Euler-Poincaré characteristic of Γ_n. Using the additivity of e we have $e(\Gamma_n) = e(\mathbb{P}^1 \times \mathcal{B}_n) + e(\Gamma_n \backslash \mathbb{P}^1 \times \mathcal{B}_n)$. By the isomorphism pr_2 on $\mathbb{P}^1 \times \mathcal{B}_n$ we get

$$\begin{aligned} e(\Gamma_n \backslash \{\mathbb{P}^1 \times \mathcal{B}_n\}) &= e(\mathbb{P}^3 \backslash \mathcal{B}_n) \\ &= e(\mathbb{P}^3) - e(\mathcal{B}_n) \\ &= 4 - e(\mathcal{B}_n), \end{aligned}$$

on the other hand

$$\begin{aligned} e(\mathbb{P}^1 \times \mathcal{B}_n) &= e(\mathbb{P}^1) \cdot e(\mathcal{B}_n) \\ &= 2 \cdot e(\mathcal{B}_n). \end{aligned}$$

Putting together

$$e(\Gamma_n) = 4 - e(\mathcal{B}_n) + 2 \cdot e(\mathcal{B}_n) = e(\mathcal{B}_n) + 4. \tag{13}$$

The reduced base locus $\mathcal{B}_n$ consists of $2n$ lines $\{L_1, \ldots, L_n\} \cup \{L'_1, \ldots, L'_n\} \subseteq Q_2$. They meet each other at n^2 points. For two intersecting lines L_i, L'_i we have

$$\begin{aligned} e(L_i \cup L'_i) &= e(L_i) + e(L'_i) - e(L_i \cap L'_i) \\ &= 2e(\mathbb{P}^1) - e(L_i \cap L'_i) \\ &= 4 - 1 \\ &= 3, \end{aligned}$$

so $e(\mathcal{B}_n) = 2n \cdot e(\mathbb{P}^1) - n^2 = 4n - n^2 = n(4-n)$. Substituting in the expression (13), we find

$$e(\Gamma_n) = n(4-n) + 4. \tag{14}$$

Consider again the first projection pr_1. Following [3] proposition 11.4, p.97, we denote by X_{gen} the generic (smooth) fibre of the map pr_1, by X_s the fibre over a point $s \in \mathbb{P}^1$ and by $s_0 = (1:0)$, s_1, s_2, s_3, $s_4 \in \mathbb{P}^1$ the critical value of pr_1, i. e. in our situation the value of $\mathbb{P}^1$ s.t. X_{s_0} is the multiple quadric Q_n and X_{s_i}, $i = 1, 2, 3, 4$, is a singular surface with nodes (in chapter 4 we showed that we have exactly 4 such surfaces in each pencil). Applying now [3], proposition 11.4, on page 97, in case of surfaces we get

$$e(\Gamma_n) = e(X_{\mathrm{gen}}) \cdot e(\mathbb{P}^1) + \sum_{i=1}^{4} (e(X_{s_i}) - e(X_{\mathrm{gen}})) + (e(Q_n) - e(X_{\mathrm{gen}})). \tag{15}$$

The calculation of $e(X_{\mathrm{gen}})$ is standard and we recall it later. We now calculate $e(X_{s_i})$. Each of the surfaces X_{s_i}, $i = 1, \ldots, 4$, has nodes, so let $\nu_i : \tilde{X}_{s_i} \longrightarrow X_{s_i}$ denote the blow up of these surfaces at the nodes. A theorem of Atiyah (cf. [1], theorem 1) shows that $e(\tilde{X}_{s_i}) = e(X_{\mathrm{gen}})$. Let now $p_1, \ldots, p_{m_i}$ be the nodes of X_{s_i} then we have

$$\begin{aligned} e(\tilde{X}_{s_i}) &= e(X_{s_i}) + \sum_{j=1}^{m_i} e(p_j) \\ &= e(X_{s_i}) + \{\text{number of nodes on } X_{s_i}\} \\ &= e(X_{s_i}) + m_i, \end{aligned}$$

so $e(X_{s_i}) = e(\tilde{X}_{s_i}) - m_i = e(X_{\mathrm{gen}}) - m_i$. Recall that $e(Q_n) = e(Q_n)_{\mathrm{red}} = e(Q_2)$, and $e(Q_2) = e(\mathbb{P}^1 \times \mathbb{P}^1) = e(\mathbb{P}^1) \cdot e(\mathbb{P}^1) = 4$. Substituting in (15), we get

$$\begin{aligned} e(\Gamma_n) &= 2e(X_{\mathrm{gen}}) + \sum_{i=1}^{4} (e(X_{\mathrm{gen}}) - m_i - e(X_{\mathrm{gen}})) + (4 - e(X_{\mathrm{gen}})) \\ &= 4 + e(X_{\mathrm{gen}}) - \sum_{i=1}^{4} m_i, \end{aligned}$$

with ν :={total number of nodes in the pencil $F_n(\lambda)$} we can write

$$e(\Gamma_n) = 4 + e(X_{\text{gen}}) - \nu. \tag{16}$$

We calculate now $e(X_{\text{gen}})$. As remarked this calculation is standard, cf. e. g. [13], p. 433-434. We will use the same notation as [13]. We write X_{gen} := X, it is a hypersurface of degree n in $\mathbb{P}^3_{\mathbb{C}}$. Consider the exact sequence of the tangent bundle T_X on X

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^3}|X \longrightarrow O_X(n) \longrightarrow 0$$

Remember that for a compact variety the second Chern class $c_2(X) = e(X)$, where by definition $c_2(X) := c_2(T_X)$. By the properties of the Chern polynomial (for the definition and the properties cf. [13], p. 429-431), we get

$$\begin{aligned} c_t(T_X)\cdot c_t(O_X(n)) &= c_t(T_{\mathbb{P}^3}|X) \\ c_t(T_X) &= \frac{c_t(T_{\mathbb{P}^3}|X)}{c_t(O_X(n))}. \end{aligned} \tag{17}$$

Let H denote the hyperplane section on X, then

$$c_t(O_X(n)) = 1 + (nH)t.$$

We have the exact Euler sequence

$$0 \longrightarrow O_{\mathbb{P}^3} \longrightarrow 4\cdot O_{\mathbb{P}^3}(1) \longrightarrow T_{\mathbb{P}^3} \longrightarrow 0,$$

and get $c_t(T_{\mathbb{P}^3}) = \dfrac{c_t(4\cdot O_{\mathbb{P}^3}(1))}{c_t(O_{\mathbb{P}^3})}$. Let h denote the hyperplane section on $\mathbb{P}^3$. We have $c_t(4\cdot O_{\mathbb{P}^3}(1)) = (1+ht)^4$ and $c_t(O_{\mathbb{P}^3}) = 1$, therefore we get

$$c_t(T_{\mathbb{P}^3}) = (1 + 4ht + 6h^2t^2 + o(t^3)).$$

Substituting in (17) and developing $\dfrac{1}{1+(nH)t}$ as Taylor series, we get

$$\begin{aligned} c_t(T_X) &= (1 + 4Ht + 6H^2t^2 + o(t^3))(1 - nHt + n^2H^2t^2 + o(t^3)) \\ c_2(T_X) &= (6 - 4n + n^2)H^2 \end{aligned}$$

Since $H^2 = H\cdot H = n$ we find $e(X) = c_2(T_X) = (6-4n+n^2)n$.
Putting this value in (16) we get

$$e(\Gamma_n) = 4 + n(6 - 4n + n^2) - \nu.$$

Using (14) we find

$$\begin{aligned} n(4-n)+4 &= 4+n(6-4n+n^2)-\nu, \\ \nu &= n(n^2-3n+2). \end{aligned}$$

For $n = 6$, 8, 12, we find the following total number of nodes in $F_6(\lambda)$, $F_8(\lambda)$, $F_{12}(\lambda)$:

n	6	8	12
ν	120	336	1320

They are in fact the same numbers, S, which we found in chapter 4 (cf. table on page 55).

5.2 Morse theory on the pencils $F_n(\lambda)$

We have seen that each point of $\mathbb{P}^3_{\mathbb{C}}$, outside the base locus, determines a surface in the pencil $F_n(\lambda)$. We consider just the real points, i.e. the points of $\mathbb{P}^3_{\mathbb{R}}$, and define a map for each $n = 6, 8, 12$:

$$\begin{aligned} \phi_n : \ \mathbb{P}^3_{\mathbb{R}} &\longrightarrow \mathbb{R} \\ x &\longmapsto -\frac{S_n(x)}{Q_n(x)} \end{aligned}$$

In fact a point $p \in \mathbb{P}^3_{\mathbb{R}}$ belongs to the surface $S_n(x) - \dfrac{S_n(p)}{Q_n(p)}Q_n(x) = 0$. Since $Q_n(x) \neq 0$ for all $x \in \mathbb{P}^3_{\mathbb{R}}$, the map ϕ_n is C^∞. We show that:
1) The critical points of ϕ_n are singular points on the surfaces in the pencil $F_n(\lambda)$ (for the definition of critical point cf. [20], p. 4).
2) The Hesse matrix of ϕ_n at a critical point p, up to multiplication by a positive scalar, is equal to minus the Hesse matrix of $F_n(\lambda)$ at p, $\lambda = -\frac{S_n(p)}{Q_n(p)}$.
Proof. **1)** A point $p \in \mathbb{P}^3_{\mathbb{R}}$ is a critical point of ϕ_n if and only if all the derivatives $\partial_i\phi_n$, $i = 0, 1, 2, 3$ vanish at p. We have

$$\partial_i\Big(-\frac{S_n(x)}{Q_n(x)}\Big)_{|p} = \Big(\frac{-\partial_i S_n(x)Q_n(x) + S_n(x)\partial_i Q_n(x)}{Q_n(x)^2}\Big)_{|p}$$

and these are all equal to zero if and only if p is a singular point on the surface $\{S_n(x) - \dfrac{S_n(p)}{Q_n(p)}Q_n(x) = 0\}$.

2) We consider the second derivatives $\partial_j\partial_i\phi_n$, $i,j = 0,1,2,3$. We have

$$\begin{aligned}
\partial_j\partial_i\left(\frac{S_n(x)}{Q_n(x)}\right)_{|p} &= \frac{\partial_j(-\partial_i S_n(x)Q_n(x) + S_n(x)\partial_i Q_n(x))_{|p}}{Q_n(p)^2} \\
&\quad + (\underbrace{-\partial_i S_n(x)Q_n(x) + S_n(x)\partial_i Q_n(x))_{|p}}_{=0}) \cdot \left(-\frac{2}{Q_n(p)^3}\right) \\
&= \frac{\partial_j(-\partial_i S_n(x)Q_n(x) + S_n(x)\partial_i Q_n(x))_{|p}}{Q_n(p)^2} \\
&= (-\partial_j\partial_i S_n(x)Q_n(x) - \partial_i S_n(x)\partial_j Q_n(x) \\
&\quad + \partial_j S_n(x)\partial_i Q_n(x) + S_n(x)\partial_j\partial_i Q_n(x))_{|p}\frac{1}{Q_n(p)^2}.
\end{aligned}$$

Now $\partial_i S_n(x) = -\frac{S_n(p)}{Q_n(p)}\partial_i Q_n(x)$ for all $i = 0,1,2,3$, therefore we get:

$$\partial_j\partial_i\left(\frac{S_n(x)}{Q_n(x)}\right)_{|p} = (-\partial_j\partial_i S_n(x)Q_n(x) + S_n(x)\partial_j\partial_i Q_n(x))_{|p}\frac{1}{Q_n(p)^2}.$$

Hence the Hesse matrix of $F_n(-\frac{S_n(p)}{Q_n(p)})$ at p is

$$(\partial_i\partial_j(S_n(x) - \frac{S_n(p)}{Q_n(p)}Q_n(x))_{|p}) = -Q_n(p)(\partial_j\partial_i(-\frac{S_n(x)}{Q_n(x)})_{|p}).$$

□

We have seen that the values of $\lambda \in \mathbb{P}^1$ s. t. $F_n(\lambda)$ is singular ($n = 6, 8, 12$) are all real and the singular points too, therefore these are the critical values and the critical points of ϕ_n (by **1)**). The *index* of a critical point of ϕ_n is defined as the number of negative eigenvalues of the Hesse matrix of ϕ_n at p, hence by **2)**, of minus the Hesse matrix of $F_n(-\frac{S_n(p)}{Q_n(p)})$ at p. Clearly if $p \in F_n(\lambda)$ is a critical point with index m, then all the points in the orbit of p under the action of G_n have the same index. With the help of MAPLE it is possible to calculate the index of each singular point (cf. chapter 4, pages 54-55, where we calculated the Hesse matrix of $F_6(-1)$, $F_8(-1)$ and $F_{12}(0)$ at $\mathbb{I}$). In the following table we give the degree n of the pencil, the value λ for which $F_n(\lambda)$ is singular, the index of the singular points of $F_n(\lambda)$ and the number C_i of critical points with this index (cf. table on page 55):

n					
6	λ	-1	$-2/3$	$-7/12$	$-1/4$
	index	0	1	2	3
	C_i	12	48	48	12
8	λ	-1	$-3/4$	$-9/16$	$-5/9$
	index	0	1	2	3
	C_i	24	72	144	96
12	λ	$-3/32$	$-22/243$	$-2/25$	0
	index	0	1	2	3
	C_i	300	600	360	60

A theorem of Morse (cf. [20], p. 29, theorem 5.2) gives the following formula for the Euler Poincaré characteristic of $\mathbb{P}^3_{\mathbb{R}}$(we use the notation of [20]):

$$\chi(\mathbb{P}^3_{\mathbb{R}}) = \sum_{i=0}^{3}(-1)^i C_i = C_0 - C_1 + C_2 - C_3. \tag{18}$$

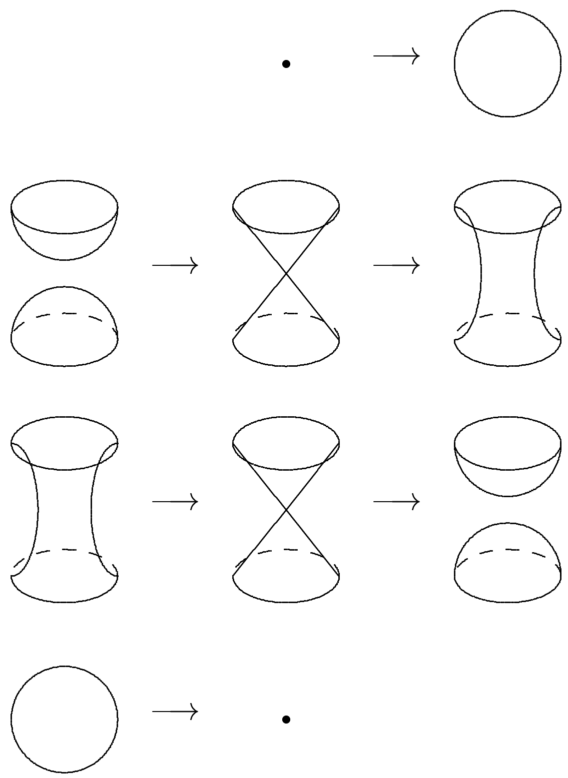
In fact this formula holds when given $a \in \mathbb{R}$, $\phi_n^{-1}(a)$ contains at most one critical point, which is not the case here. In our situation on a fiber $\phi_n^{-1}(a)$, $a \in \mathbb{R}$, the critical points (if there are) have all the same index, so using remark 3.3, p. 19 of [20], we see that the theorem holds in this case too. Substituting the values of the table above on the right hand side of the equality (18), we get

$$\begin{aligned} n &= 6: & 12 - 48 + 48 - 12 &= 0 \\ n &= 8: & 24 - 72 + 144 - 96 &= 0 \\ n &= 12: & 300 - 600 + 360 - 60 &= 0, \end{aligned}$$

this agrees with the well known fact $\chi(\mathbb{P}^3_{\mathbb{R}}) = 0$ (cf. e.g. [10], p. 100).

Using now the local equation of ϕ_n at a critical point p (cf. [20], p. 6, lemma 2.2), it is possible to show how the surfaces of the pencils behave close to p. In the following picture we write on the left border the index of a critical point p, then the local equation of ϕ_n at p. This is 0 at p, so the fourth column shows how the surface which contains p looks like at p in $\mathbb{P}^3_{\mathbb{R}}$. For $c < 0$ or $c > 0$ we see how the other surfaces of the pencil behave close to p.

index	local equation	$c < 0$	$c = 0$	$c > 0$
0	$x^2 + y^2 + z^2 = c$			
1	$x^2 + y^2 - z^2 = c$			
2	$-x^2 - y^2 + z^2 = c$			
3	$-x^2 - y^2 - z^2 = c$			



6 Tables

We choose a represent σ in each conjugacy class of the table on page 40 and we write it in the first column of the following tables with its order. Then we give the number N_1 of fix lines of the elements in its conjugacy class, and the fix line(s) of the chosen matrix σ. In the third column we write the surfaces of $F_n(\lambda)$, which the fix line(s) of $[\sigma]$ meets at singular points. Then we give the singular points, which are also the ramification points of a morphism (9) or (10) and eventually the intersections with the base locus. Finally we write the configuration of the fix lines of the elements in $[\sigma]$ with the singular points of the surfaces in the pencil. We put $a := 1+\sqrt{2}$ and $\tau := \frac{1}{2}(1+\sqrt{5})$.

6.1 Bi-tetrahedral group G_6

Matrix	N_1	Fix line(s)	Surface	Fix points	Config.
σ_{24} order 4	18	$\langle \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \rangle$	$F_6(-1)$	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$(18_2, 12_3)$
			$F_6(-\frac{1}{4})$	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$	$(18_2, 12_3)$
				+2 points in the base locus	
		$\langle \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \rangle$	$F_6(-1)$	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$(18_2, 12_3)$
			$F_6(-\frac{1}{4})$	$\begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\1\end{pmatrix}$	$(18_2, 12_3)$
				+2 points in the base locus	

Matrix	N_1	Fix line(s)	Surface	Fix points	Config.
$\pi_3\pi_3'$ $(\pi_3^2\pi_3'^2)$ order 3	16	$\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\1\end{pmatrix} \rangle$	$F_6(-\frac{2}{3})$	$\begin{pmatrix}3\\1\\-1\\1\end{pmatrix}, \begin{pmatrix}3\\-1\\1\\-1\end{pmatrix}$ $\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}$	$(16_3, 48_1)$
			$F_6(-1)$	$\begin{pmatrix}1\\1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}$ $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$(16_3, 12_4)$
$\pi_3\pi_3'^2$ $(\pi_3^2\pi_3')$ order 6	16	$\langle \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\1\end{pmatrix} \rangle$	$F_6(-\frac{7}{12})$	$\begin{pmatrix}0\\2\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\\2\end{pmatrix}$ $\begin{pmatrix}0\\1\\2\\1\end{pmatrix}$	$(16_3, 48_1)$
			$F_6(-\frac{1}{4})$	$\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\1\end{pmatrix}$ $\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$	$(16_3, 12_4)$

6.2 Bi-octahedral group G_8

Matrix	N_1	Fix line(s)	Surface	Fix points	Config.
σ_{24} order 4	18	$\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rangle$	$F_8(-1)$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \end{pmatrix}$	$(18_4, 24_3)$
			$F_8(-\frac{3}{4})$	$\begin{pmatrix} 1 \\ 0 \\ \pm a \\ 0 \end{pmatrix}, \begin{pmatrix} \pm a \\ 0 \\ 1 \\ 0 \end{pmatrix}$	$(18_4, 72_1)$
		$\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle$	$F_8(-1)$	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \pm 1 \end{pmatrix}$	$(18_4, 24_3)$
			$F_8(-\frac{3}{4})$	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \pm a \end{pmatrix}, \begin{pmatrix} 0 \\ \pm a \\ 0 \\ 1 \end{pmatrix}$	$(18_4, 72_1)$
$\pi_3\pi_3'$ order 3	32	$\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \rangle$	$F_8(-1)$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$	$(32_3, 24_4)$
			$F_8(-\frac{5}{9})$	$\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} \pm 3 \\ 1 \\ -1 \\ 1 \end{pmatrix}$	$(32_3, 96_1)$
				+2 points in the base locus	

Matrix	N_1	Fix line(s)	Surface	Fix points	Config.
$\pi_4\pi_4'$ order 4	18	$\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \rangle$	$F_8(-1)$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $\begin{pmatrix}1\\\pm1\\0\\0\end{pmatrix}$	$(18_4, 24_3)$
			$F_8(-\frac{3}{4})$	$\begin{pmatrix}1\\\pm a\\0\\0\end{pmatrix}, \begin{pmatrix}\pm a\\1\\0\\0\end{pmatrix}$	$(18_4, 72_1)$
$\pi_3\pi_4\pi_3'\pi_4'$ order 2	72	$\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix} \rangle$	$F_8(-1)$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}$	$(72_2, 24_6)$
			$F_8(-\frac{5}{9})$	$\begin{pmatrix}\pm1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}\pm2\\1\\0\\1\end{pmatrix}$	$(72_4, 96_3)$
			$F_8(-\frac{9}{16})$	$\begin{pmatrix}\pm\sqrt{2}\\1\\0\\1\end{pmatrix}$	$(72_2, 144_1)$
		$\langle \begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \rangle$	$F_8(-1)$	$\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$(72_2, 24_6)$
			$F_8(-\frac{5}{9})$	$\begin{pmatrix}0\\1\\\pm1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\\pm2\\-1\end{pmatrix}$	$(72_4, 96_3)$
			$F_8(-\frac{9}{16})$	$\begin{pmatrix}0\\1\\\pm\sqrt{2}\\-1\end{pmatrix}$	$(72_2, 144_1)$

Matrix	N_1	Fix line(s)	Surface	Fix points	Config.
$\pi_3\pi_4\sigma_4$ order 2	36	$\langle \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \rangle$	$F_8(-\frac{9}{16})$	$\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ -1 \\ -\sqrt{2} \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ 1 \\ 0 \\ -1 \end{pmatrix}$	$(36_4, 144_1)$
			$F_8(-\frac{3}{4})$	$\begin{pmatrix} a \\ 1 \\ -1 \\ -a \end{pmatrix}, \begin{pmatrix} 1 \\ a \\ a \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \\ -a \\ -a \end{pmatrix}, \begin{pmatrix} a \\ a \\ 1 \\ -1 \end{pmatrix}$	$(36_4, 72_2)$
		$\langle \begin{pmatrix} \sqrt{2} \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \\ 0 \end{pmatrix} \rangle$	$F_8(-\frac{9}{16})$	$\begin{pmatrix} \sqrt{2} \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ -\sqrt{2} \end{pmatrix}$	$(36_4, 144_1)$
			$F_8(-\frac{3}{4})$	$\begin{pmatrix} a \\ -a \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -a \\ a \end{pmatrix}$ $\begin{pmatrix} a \\ -1 \\ -1 \\ a \end{pmatrix}, \begin{pmatrix} -1 \\ a \\ -a \\ 1 \end{pmatrix}$	$(36_4, 72_2)$

Matrix	N_1	Fix line(s)	Surface	Fix points	Config.
$\sigma_2\pi_3'\pi_4'$ order 2	36	$\langle \begin{pmatrix} \sqrt{2} \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ \sqrt{2} \\ 1 \\ 0 \end{pmatrix} \rangle$	$F_8(-\frac{9}{16})$	$\begin{pmatrix} \sqrt{2} \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ \sqrt{2} \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ \sqrt{2} \end{pmatrix}$	$(36_4, 144_1)$
			$F_8(-\frac{3}{4})$	$\begin{pmatrix} 1 \\ 1 \\ a \\ a \end{pmatrix}, \begin{pmatrix} a \\ -a \\ -1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} a \\ -1 \\ 1 \\ a \end{pmatrix}, \begin{pmatrix} -1 \\ a \\ a \\ 1 \end{pmatrix}$	$(36_4, 72_2)$
		$\langle \begin{pmatrix} 0 \\ 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ -\sqrt{2} \end{pmatrix} \rangle$	$F_8(-\frac{9}{16})$	$\begin{pmatrix} 0 \\ 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ -\sqrt{2} \end{pmatrix}$ $\begin{pmatrix} \sqrt{2} \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \\ 0 \end{pmatrix}$	$(36_4, 144_1)$
			$F_8(-\frac{3}{4})$	$\begin{pmatrix} a \\ a \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ a \\ -a \end{pmatrix}$ $\begin{pmatrix} a \\ 1 \\ 1 \\ -a \end{pmatrix}, \begin{pmatrix} 1 \\ a \\ -a \\ -1 \end{pmatrix}$	$(36_4, 72_2)$

6.3 Bi-icosahedral group G_{12}

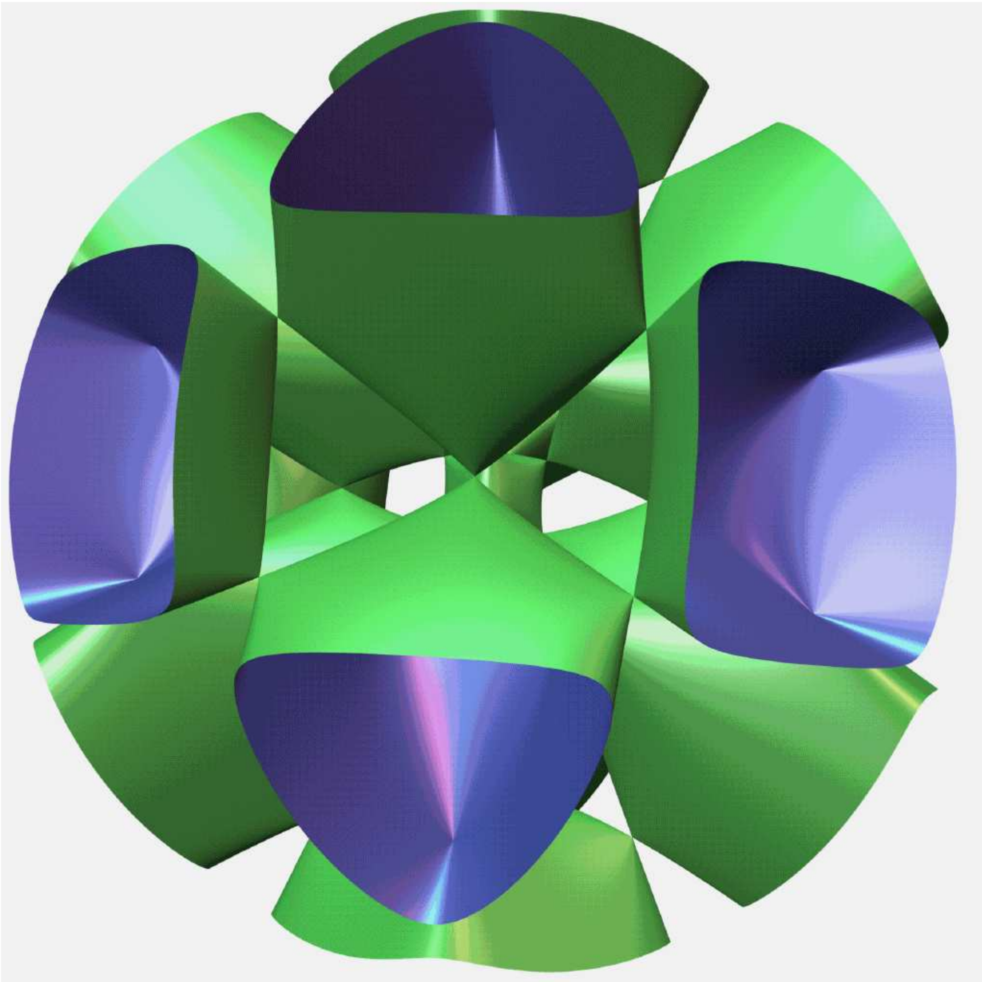
Matrix	N_1	Fix line(s)	Surface	Fix points	Config.
σ_{24} order 4	450	$\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle$	$F_{12}(-\frac{3}{32})$	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$	$(450_2, 300_3)$
			$F_{12}(-\frac{2}{25})$	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \pm\tau \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \pm(1-\tau) \end{pmatrix}$	$(450_4, 360_5)$
			$F_{12}(0)$	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$(450_2, 60_{15})$
			$F_{12}(-\frac{22}{243})$	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \pm(\tau+1) \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \\ \pm(2-\tau) \end{pmatrix}$	$(450_4, 600_3)$

Matrix	N_1	Fix line(s)	Surface	Fix points	Config.
σ_{24} order 4	450	$\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rangle$	$F_{12}(-\frac{3}{32})$	$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$	$(450_2, 300_3)$
			$F_{12}(-\frac{2}{25})$	$\begin{pmatrix} 1 \\ 0 \\ \pm\tau \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \pm(1-\tau) \\ 0 \end{pmatrix}$	$(450_4, 360_5)$
			$F_{12}(0)$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$	$(450_2, 60_{15})$
			$F_{12}(-\frac{22}{243})$	$\begin{pmatrix} 1 \\ 0 \\ \pm(\tau+1) \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ \pm(2-\tau) \\ 0 \end{pmatrix}$	$(450_4, 600_3)$

Matrix	N_1	Fix line(s)	Surface	Fix points	Config.
$\pi_5^2\sigma_2\pi_5'^2\sigma_4$ order 3	200	$\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \tau-2 \\ 0 \end{pmatrix} \rangle$	$F_{12}(0)$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} \pm(1-\tau) \\ 1 \\ \tau-2 \\ 0 \end{pmatrix}$	$(200_3, 60_{10})$
			$F_{12}(-\frac{22}{243})$	$\begin{pmatrix} 0 \\ 1 \\ \tau-2 \\ 0 \end{pmatrix}$ $\begin{pmatrix} \pm3(1-\tau) \\ 1 \\ \tau-2 \\ 0 \end{pmatrix}$	$(200_3, 600_1)$
			$F_{12}(-\frac{3}{32})$	$\begin{pmatrix} \pm(\tau+1) \\ 1 \\ \tau-2 \\ 0 \end{pmatrix}$ $\begin{pmatrix} \pm(3-\tau) \\ 1 \\ \tau-2 \\ 0 \end{pmatrix}$ $\begin{pmatrix} \pm(5-3\tau) \\ 1 \\ \tau-2 \\ 0 \end{pmatrix}$	$(200_6, 300_4)$

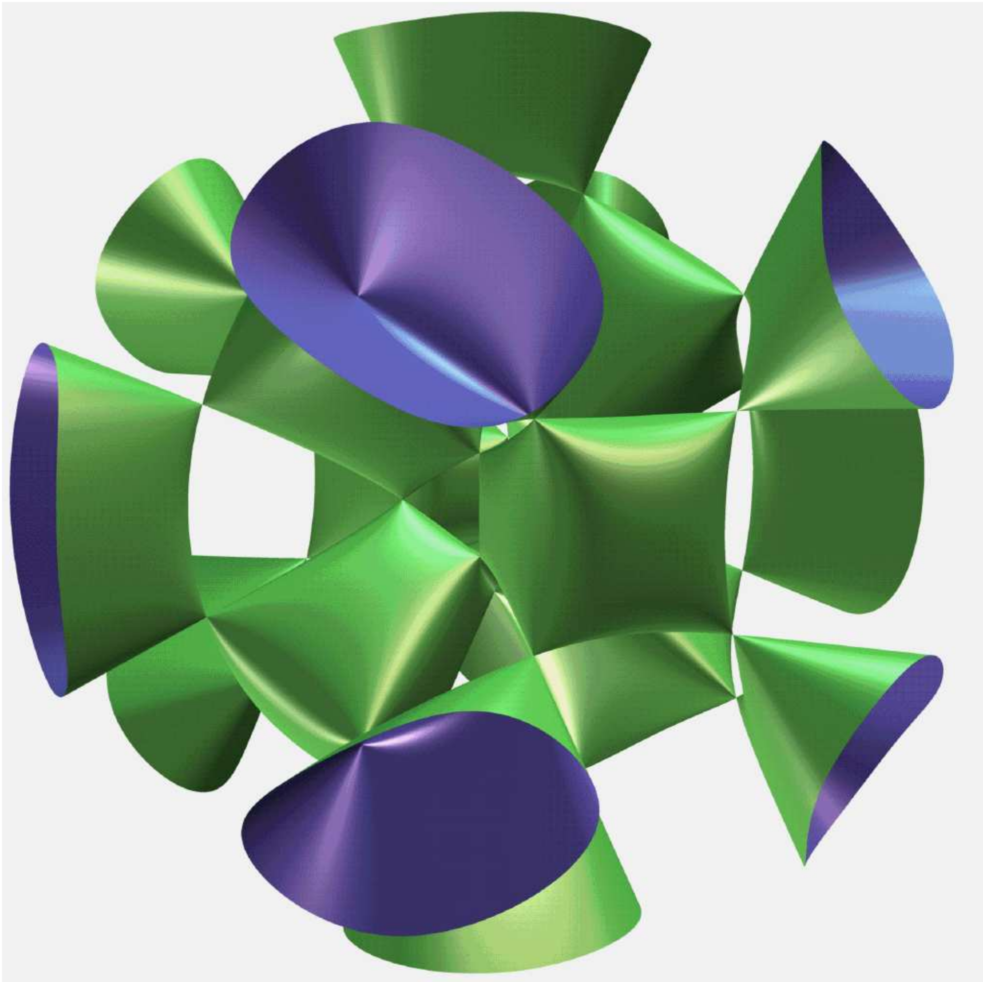
Matrix	N_1	Fix line(s)	Surface	Fix points	Config.
$\pi_5\pi_5'$ $(\pi_5^2\pi_5'^2)$ order 5	72	$\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \tau-1 \\ 1 \end{pmatrix} \right\rangle$	$F_{12}(-\frac{2}{25})$	$\begin{pmatrix} 0 \\ 0 \\ \tau-1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} \pm(\tau+2) \\ 0 \\ \tau-1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} \pm(4-3\tau) \\ 0 \\ \tau-1 \\ 1 \end{pmatrix}$	$(72_5, 360_1)$
			$F_{12}(0)$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} \pm(2-\tau) \\ 0 \\ \tau-1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} \pm\tau \\ 0 \\ \tau-1 \\ 1 \end{pmatrix}$	$(72_5, 60_6)$
				+2 points in the base locus	

7 Pictures



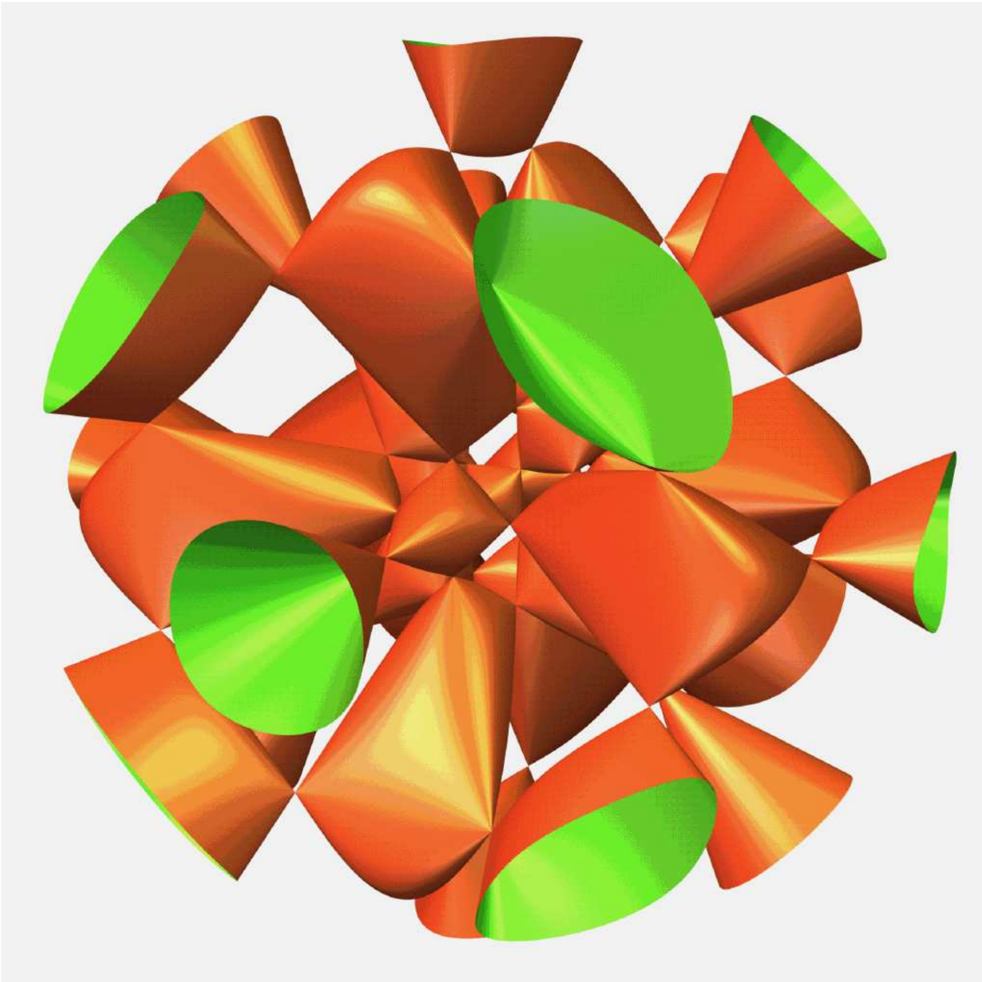
$A_4 \times A_4$–symmetric sextic with 48 nodes

$$x_0^6 + x_1^6 + x_2^6 + x_3^6 + 15\left(x_0^2x_1^2x_2^2 + x_0^2x_1^2x_3^2\right) +$$
$$+15\left(x_0^2x_2^2x_3^2 + x_1^2x_2^2x_3^2\right) - \frac{2}{3}(x_0^2 + x_1^2 + x_2^2 + x_3^2)^3 = 0$$



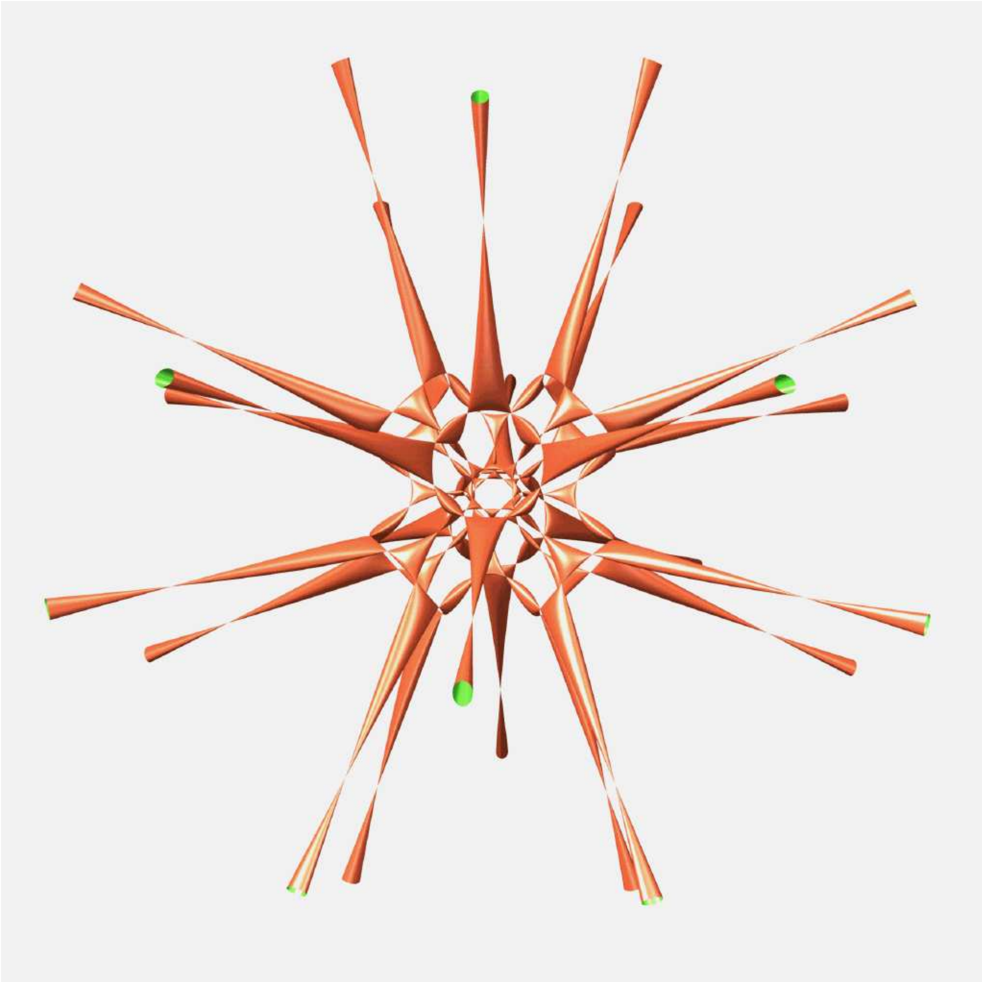
$A_4 \times A_4$–symmetric sextic with 48 nodes

$$x_0^6 + x_1^6 + x_2^6 + x_3^6 + 15\left(x_0^2x_1^2x_2^2 + x_0^2x_1^2x_3^2\right) +$$
$$+15\left(x_0^2x_2^2x_3^2 + x_1^2x_2^2x_3^2\right) - \tfrac{7}{12}(x_0^2 + x_1^2 + x_2^2 + x_3^2)^3 = 0$$



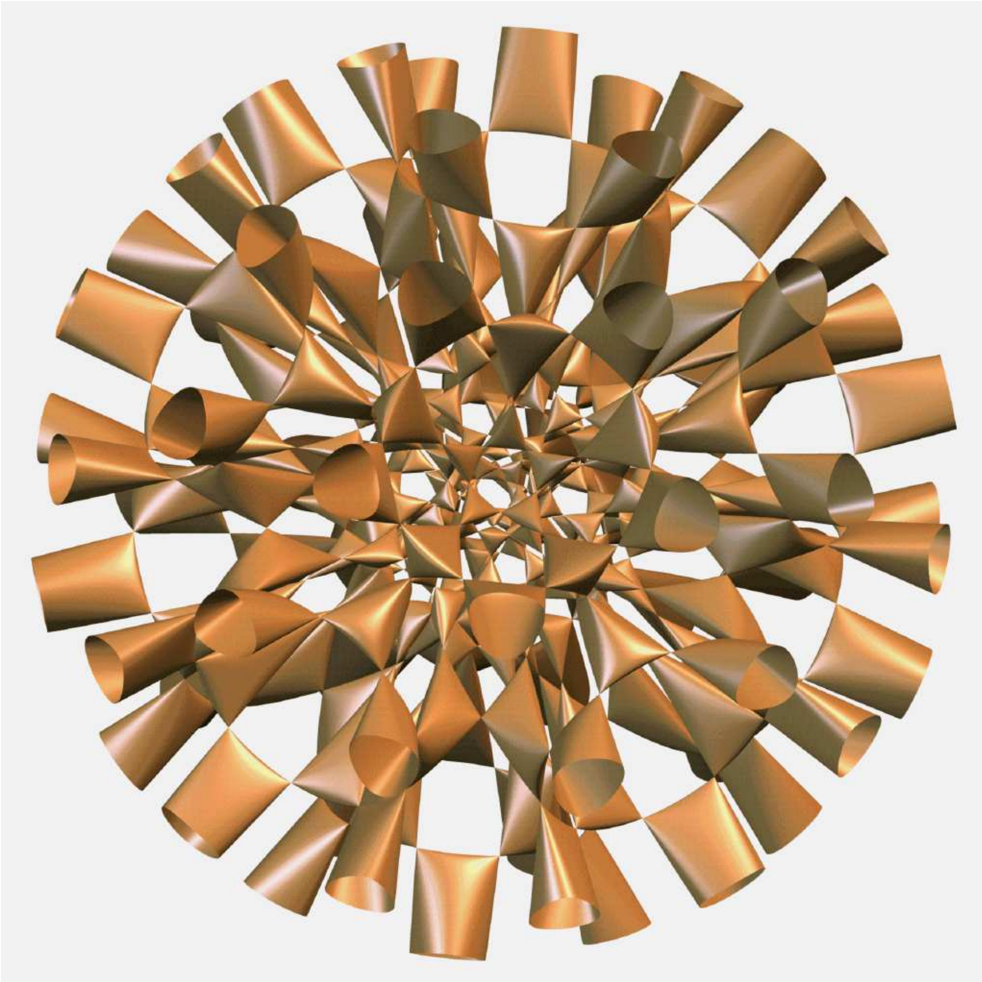
$S_4 \times S_4$–symmetric octic with 72 nodes

$$x_0^8+x_1^8+x_2^8+x_3^8+14\left(x_0^4x_1^4+x_0^4x_2^4+x_0^4x_3^4+x_1^4x_2^4+x_1^4x_3^4+x_2^4x_3^4\right)+$$
$$+168x_0^2x_1^2x_2^2x_3^2-\tfrac{3}{4}(x_0^2+x_1^2+x_2^2+x_3^2)^4=0$$



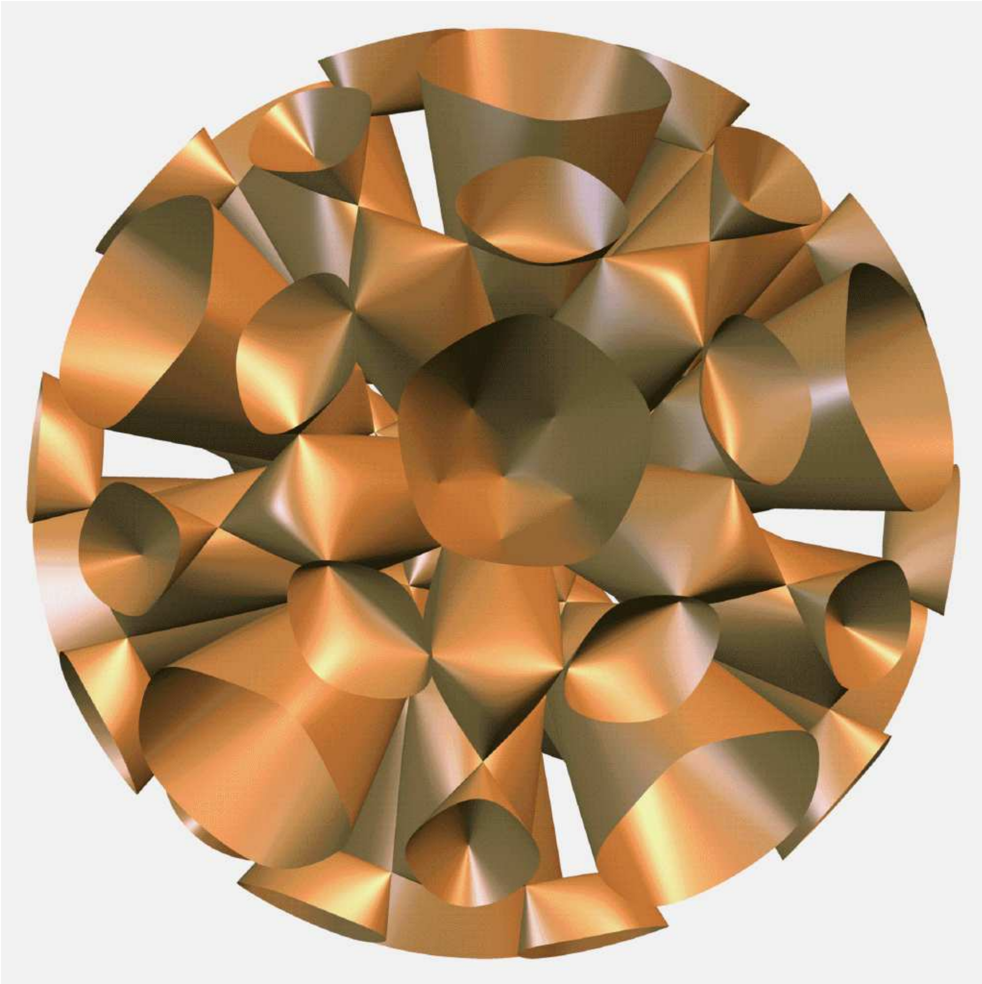
$S_4 \times S_4$–symmetric octic with 144 nodes

$$x_0^8+x_1^8+x_2^8+x_3^8+14\left(x_0^4x_1^4+x_0^4x_2^4+x_0^4x_3^4+x_1^4x_2^4+x_1^4x_3^4+x_2^4x_3^4\right)+$$
$$+168x_0^2x_1^2x_2^2x_3^2-\tfrac{9}{16}(x_0^2+x_1^2+x_2^2+x_3^2)^4=0$$



$A_5 \times A_5$–symmetric surface of degree 12

with 600 nodes



$A_5 \times A_5$–symmetric surface of degree 12

with 360 nodes

Notations

$\mathrm{SO}(n)$	special orthogonal group, 1
V	Klein four group, 1
A_1, A_2, A_3	rotations of order two in V, 1
T	tetrahedral group, 1
A_4	even permutation group of four objects, 2
R_3	rotation of order three in T, 2
O	octahedral group, 2
S_4	permutation group of four objects, 3
R_4	rotation of order four in O, 3
I	icosahedral group, 4
A_5	even permutation group of five objects, 4
$\tau := \frac{1}{2}(1+\sqrt{5})$	golden section number, 4
R_5	rotation of order five in I, 4
$\mathbb{H}$	real algebra of Hamilton's quaternions, 5
q_0, q_1, q_2, q_3	basis of $\mathbb{H}$, 5
$\mathrm{SU}(2)$	special unitary group, 6
$\rho : \mathrm{SU}(2) \to \mathrm{SO}(3)$	2:1 morphism, 6
$\mathbb{I}$	identity matrix, 6
$\tilde{G}$	binary group, 6
$\tilde{V} := \rho^{-1}(V)$	binary Klein four group, 6
$\tilde{A}_4 := \rho^{-1}(T)$	binary tetrahedral group, 7
p_3	matrix of order six in $\tilde{A}_4$, 7
$\tilde{S}_4 := \rho^{-1}(O)$	binary octahedral group, 7
p_4	matrix of order eight in $\tilde{S}_4$, 7
$\tilde{A}_5 := \rho^{-1}(I)$	binary icosahedral group, 7
p_5	matrix of order ten in $\tilde{A}_5$, 7
$\sigma : \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$	2:1 morphism, 7
$\mathcal{H} := \sigma(\tilde{V} \times \tilde{V})$	Heisenberg group, 8
$\sigma_1 := \sigma(q_1, \mathbb{I}), \sigma_2 := \sigma(q_2, \mathbb{I})$, $\sigma_3 := \sigma(\mathbb{I}, q_1), \sigma_4 := \sigma(\mathbb{I}, q_2)$	matrices of order four in $\mathcal{H}$, 8
$G_6 := \sigma(\tilde{A}_4 \times \tilde{A}_4)$	bi-tetrahedral group, 9
$\pi_3 := \sigma(p_3, \mathbb{I}), \pi_3' := \sigma(\mathbb{I}, p_3)$	matrices of order six in G_6, 9
$G_8 := \sigma(\tilde{S}_4 \times \tilde{S}_4)$	bi-octahedral group, 9
$\pi_4 := \sigma(p_4, \mathbb{I}), \pi_4' := \sigma(\mathbb{I}, p_4)$	matrices of order eight in G_8, 9
$G_{12} := \sigma(\tilde{A}_5 \times \tilde{A}_5)$	bi-icosahedral group, 9
$\pi_5 := \sigma(p_5, \mathbb{I}), \pi_5' := \sigma(\mathbb{I}, p_5)$	matrices of order ten in G_{12}, 10
$\mathbb{C}[x_0, x_1, x_2, x_3]$	complex polynomial ring in four variables, 13
C	matrix of order two in $\mathrm{GL}(4, \mathbb{R})$, 17

σ_{24}	product of σ_2 and σ_4, 19
$\mathbb{C}[x_0, x_1, x_2, x_3]_j^G$	$\mathbb{C}$-vector space of G-invariant homogeneous polynomials of degree j, 22
$Q_j(x)$	complex multiple quadric $(x_0^2 + x_1^2 + x_2^2 + x_3^2)^{\frac{j}{2}}$, 26
$S_6(x)$	invariant polynomial under G_6, 27
$S_8(x)$	invariant polynomial under G_8, 28
$S_{12}(x)$	invariant polynomial under G_{12}, 30
$F_n(\lambda)$	pencil $S_n(x) + \lambda Q_n(x) = 0$, 33
S_n,Q_n	surfaces which correspond to the sets $\{S_n(x) = 0\}$, $\{Q_n(x) = 0\}$, 33
$\mathbb{P}$GL(2)	projectivization of the space of invertible complex 2×2-matrices, 34
$\mathcal{B}_n$	intersection of Q_2 and S_n, 35

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