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The Classification of Hypersurfaces of a Euclidean Space with Parallel Higher Order Fundamental Form

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Introduction

Submanifolds with parallel second fundamental form are defined as an extrinsic analogue of locally symmetric manifolds. From the classification of submanifolds with parallel second fundamental form in a Euclidean space by D. Ferus in [F] it follows that all of them are locally invariant under the reflection in the normal space of an arbitrary point. This property was proved directly later on by W. Strübing in [S] without using the classification of Ferus. As a consequence, submanifolds with parallel second fundamental form, also called parallel submanifolds, of a Euclidean space are locally symmetric. This follows of course also immediately from the equation of Gauss.

It was proved by K. Nomizu that $\nabla^k R=0$ implies $\nabla R=0$ for Riemannian manifolds (for the global version that requires completeness, see [N−O]). One could ask whether this is also the case for the higher derivatives of the second fundamental form, which we call the higher fundamental forms. At this moment, not very much is known about submanifolds for which some higher order fundamental form is parallel, or say higher order parallel submanifolds. In $[L]_1$ Ü. Lumiste studies flat submanifolds of a Euclidean space with flat normal connection and parallel third fundamental form. As an example he mentions the Cornu spiral, which is a plane curve whose curvature is proportional to the arc length. This example also shows that $\nabla^k h=0$ does not imply $\nabla h=0$, since $\nabla^2 h=0$ for the Cornu spiral, but $\nabla h \neq 0$. In $[L]_2$, the same author classifies two-codimensional submanifolds and surfaces of a Euclidean space with parallel third fundamental form. As a consequence, one can give a classification of hypersurfaces of a Euclidean space that satisfy $\nabla^2 h=0$.

In this paper we classify hypersurfaces M^n of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with parallel higher order fundamental form, i.e. that satisfy $\nabla^k h=0$ for some k, where h is the second fundamental form of M^n. In particular we show the following theorem:

Theorem. *Let M^n be a hypersurface of $\mathbb{R}^{n+1}$ such that $\nabla^k h=0$ for some k. Then M^n is an open part of one of the following hypersurfaces:*

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(1) *an affine hyperplane* $\mathbb{R}^n$,

(2) *a hypersphere* S^n,

(3) *a product of an affine subspace* $\mathbb{R}^{n-m}$ *and a sphere* S^m *in a* $(m+1)$*-dimensional affine subspace* $\mathbb{R}^{m+1}$, *orthogonal to* $\mathbb{R}^{n-m}$,

(4) *a cylinder on a plane curve whose curvature function is a polynomial function of degree at most* $k-1$ *of the arc length.*

Examples (1), (2) and (3) are parallel, thus satisfying $\nabla h=0$, but Example (4) is only parallel if the curve on which the cylinder is built, is either a circle or a straight line, i.e. a curve whose curvature is a polynomial function of degree zero of the arc length.

We call plane curves whose curvature is a polynomial function of the arc length "polynomial spirals". In the last section of this paper we show some pictures of this kind of curves.

§ 1. Preliminaries

Let M^n be an immersed hypersurface of the Euclidean space $\mathbb{R}^{n+1}$. We denote the Euclidean metric on $\mathbb{R}^{n+1}$ by $\langle\ ,\ \rangle$ and the Levi Civita connection of $\langle\ ,\ \rangle$ by D. The induced metric on M^n is also denoted by $\langle\ ,\ \rangle$ and the Levi Civita connection of $(M^n, \langle\ ,\ \rangle)$ by ∇. Then we have the formulas of Gauss and Weingarten.

$$D_X Y=\nabla_X Y+h(X, Y)\,\xi,$$
$$D_X \xi=-SX,$$

whereby X and Y are tangent vector fields, ξ is a unit normal vector field and h and S denote respectively the second fundamental form and the shape operator of M^n. Then h and S are related by

$$\langle SX, Y\rangle=h(X, Y).$$

Since h is symmetric it follows that S is symmetric, and therefore there exists an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of $T_p M$ for every $p\in M^n$ consisting of eigenvectors of S, i.e. $S e_i=\lambda_i e_i$. The numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the principal curvatures of M^n at p. If a basis $\{e_1, e_2, \ldots, e_n\}$ occurs in the following, it will always mean a basis of eigenvectors of S. Then the equation of Gauss states that

$$R(e_i, e_j)=\lambda_i \lambda_j e_i \wedge e_j, \tag{1.1}$$

whereby Λ associates to two vectors $X, Y\in T_p M$ an endomorphism $X\wedge Y$ of $T_p M$ by

$$(X\wedge Y)\,Z=\langle Y, Z\rangle X-\langle X, Z\rangle Y.$$

According to Lemma 2.1 in [R] there exist continuous functions $\lambda_1, \lambda_2, \ldots, \lambda_n$ on M^n, such that for every $p\in M^n$ $\lambda_1(p), \lambda_2(p), \ldots, \lambda_n(p)$ are the eigenvalues of S.

M^n is called totally geodesic if $h=0$. It is well known that M^n is totally geodesic if and only if M^n is an open part of a hyperplane. M^n is called totally umbilical if $\lambda_1=\lambda_2=\ldots=\lambda_n=\lambda$. It is well known that M^n is totally umbilical if and only if M^n is totally geodesic ($\lambda=0$), or M^n is an open part of a hypersphere ($\lambda\neq 0$). M^n is called cylindrical if rank $S\leqq 1$ in each point. M^n is cylindrical if and only if M^n is flat. The famous Hartman-Nirenberg theorem [H−N] states that a complete cylindrical hypersurface is a cylinder over a plane curve.

The k-th derivative $\nabla^k h$ of h is defined recursively by

$$(\nabla^k h)(X_1,X_2,\ldots,X_{k+2})=X_1(\nabla^{k-1}h)(X_2,\ldots,X_{k+2})$$
$$-\sum_{i=2}^{k+2}(\nabla^{k-1}h)(X_2,\ldots,\nabla_{X_1}X_i,\ldots,X_{k+2}).$$

We call $\nabla^k h$ the $(k+2)$-nd fundamental form. The Ricci identity states for $k\geqq 2$ that

$$(\nabla^k h)(X_1,X_2,\ldots,X_{k+2})-(\nabla^k h)(X_2,X_1,\ldots,X_{k+2})$$
$$=(R(X_1,X_2)\cdot(\nabla^{k-2}h))(X_3,\ldots,X_{k+2}), \tag{1.2}$$

whereby $X_1,X_2,\ldots,X_{k+2}\in\mathscr{X}(M^n)$, and $R(X_1,X_2)\cdot(\nabla^{k-2}h)$ is defined by

$$(R(X_1,X_2)\cdot(\nabla^{k-2}h))(X_3,\ldots,X_{k+2})$$
$$=-\sum_{i=3}^{k+2}(\nabla^{k-2}h)(X_3,\ldots,R(X_1,X_2)X_i,\ldots,X_{k+2}). \tag{1.3}$$

For $k=1$ the equation of Codazzi states that

$$(\nabla h)(X,Y,Z)=(\nabla h)(Y,X,Z), \tag{1.4}$$

or equivalently

$$(\nabla_Y S)(X)=(\nabla_X S)(Y).$$

If $\nabla h=0$, then M^n is called parallel, or in some papers "symmetric" or "extrinsic locally symmetric". If M^n is parallel then M^n is an open part of a hypersphere, or an open part of a hyperplane, or an open part of a product of an affine subspace $\mathbb{R}^{n-k}$ and a sphere S^k in a $(k+1)$-dimensional affine subspace $\mathbb{R}^{k+1}$, orthogonal to $\mathbb{R}^{n-k}$, see [S−W]. For a classification of parallel submanifolds M^n of $\mathbb{R}^{n+p}$, which can be defined in a similar way, see [F].

If $R\cdot h=0$, i.e. $R(X,Y)\cdot h=0$ for all X and Y, then we call M^n semi-parallel [D]. If M^n is parallel, then M^n is also semi-parallel. The semi-parallel hypersurfaces of $\mathbb{R}^{n-1}$ are classified in [D]. If M^n is semi-parallel, then M^n is an open part of a hypersphere, *or* an open part of a hyperplane, *or* an open part of an elliptic hypercone, *or* an open part of a product of an affine subspace $\mathbb{R}^{n-k}$ and a sphere S^k or an elliptic hypercone $\mathscr{C}^k$ in a $(k+1)$-dimensional affine subspace $\mathbb{R}^{k+1}$, orthogonal to $\mathbb{R}^{n-k}$, *or* else M^n is cylindrical.

Similar to (1.3) we define an operator $R\cdot$ that acts on a t-covariant tensor field by

$$(R\cdot T)(X_1, Y_1, Z_1, Z_2, \ldots, Z_t) = (R(X_1, Y_1)\cdot T)(Z_1, Z_2, \ldots, Z_t)$$
$$= -T(R(X_1, Y_1)Z_1, Z_2, \ldots, Z_t) - \ldots - T(Z_1, Z_2, \ldots, R(X_1, Y_1)Z_t).$$

Then $R\cdot T$ is a $(t+2)$-covariant tensor field. We also define recursively $R^k\cdot T$ by

$$(R^k\cdot T)(X_1, Y_1, \ldots, X_k, Y_k, Z_1, Z_2, \ldots, Z_t) = (R(X_1, Y_1)$$
$$\cdot (R^{k-1}\cdot T))(X_2, Y_2, \ldots, X_k, Y_k, Z_1, Z_2, \ldots, Z_t).$$

Then $R^k\cdot T$ is a $(t+2k)$-covariant tensor field.

§ 2. Reduction of the Condition

We now try to reduce the condition $\nabla^k h = 0$ to conditions that are more easy to handle. The following lemmas do the trick.

Lemma 2.1. *If* $\nabla^k h = 0$, *then* $R^{\left[\frac{k+1}{2}\right]}\cdot h = 0$.

Proof. If $\nabla^{2k} h = 0$ then it follows from (1.2) that $R\cdot(\nabla^{2k-2}h) = 0$. From this is follows again from (1.2) that $R\cdot(R\cdot(\nabla^{2k-4}h)) = 0$. Proceeding like this we obtain that $R^k\cdot h = 0$.

If $\nabla^{2k+1}h = 0$ then $\nabla^{2k+2}h = 0$. Then it follows like above that $R^{k+1}\cdot h = 0$. □

Lemma 2.2. (a) $(R^{2k}\cdot h)(e_i, e_j, e_i, e_j, \ldots, e_i, e_i) = (-1)^k\, 2^{2k-1}\lambda_i^{2k}\lambda_j^{2k}(\lambda_i - \lambda_j)$,

(b) $(R^{2k}\cdot h)(e_i, e_j, e_i, e_j, \ldots, e_j, e_j) = (-1)^{k+1}\, 2^{2k-1}\lambda_i^{2k}\lambda_j^{2k}(\lambda_i - \lambda_j)$,

(c) $(R^{2k}\cdot h)(e_i, e_j, e_i, e_j, \ldots, e_i, e_j) = 0$,

(d) $(R^{2k+1}\cdot h)(e_i, e_j, e_i, e_j, \ldots, e_i, e_j) = (-1)^{k+1}\, 2^{2k}\lambda_i^{2k+1}\lambda_j^{2k+1}(\lambda_i - \lambda_j)$,

(e) $(R^{2k+1}\cdot h)(e_i, e_j, e_i, e_j, \ldots, e_i, e_i) = 0$,

(f) $(R^{2k+1}\cdot h)(e_i, e_j, e_i, e_j, \ldots, e_j, e_j) = 0$.

Proof. The proof goes by induction. First let $k=0$, then by (1.1) we obtain

$$(R(e_i, e_j)\cdot h)(e_i, e_j) = -h(R(e_i, e_j)e_i, e_j) - h(e_i, R(e_i, e_j)e_j)$$
$$= h(\lambda_i\lambda_j e_j, e_j) - h(e_i, \lambda_i\lambda_j e_i) = \lambda_i\lambda_j(\lambda_j - \lambda_i),$$
$$(R(e_i, e_j)\cdot h)(e_i, e_i) = -2h(R(e_i, e_j)e_i, e_i) = 2h(\lambda_i\lambda_j e_j, e_i) = 0,$$

and similarly

$$(R(e_i, e_j)\cdot h)(e_j, e_j) = 0.$$

The formulas (a), (b) and (c) make no sense here if $k=0$. Now suppose the lemma is true for $k-1$. Then

$$(R^{2k}\cdot h)(e_i, e_j, e_i, e_j, \ldots, e_i, e_i)$$

$$\begin{aligned}
&= -(R^{2k-1}\cdot h)(R(e_i,e_j)e_i,e_j,e_i,e_j,\dots,e_i,e_i)\\
&\quad -(R^{2k-1}\cdot h)(e_i,R(e_i,e_j)e_j,e_i,e_j,\dots,e_i,e_i)\\
&\quad -\dots-2(R^{2k-1}\cdot h)(e_i,e_j,e_i,e_j,\dots,R(e_i,e_j)e_i,e_i)\\
&=(R^{2k-1}\cdot h)(\lambda_i\lambda_j e_j,e_j,e_i,e_j,\dots,e_i,e_i)\\
&\quad -(R^{2k-1}\cdot h)(e_i,\lambda_i\lambda_j e_i,e_i,e_j,\dots,e_i,e_i)\\
&\quad -\dots+2(R^{2k-1}\cdot h)(e_i,e_j,e_i,e_j,\dots,\lambda_i\lambda_j e_j,e_i)\\
&=\lambda_i\lambda_j(R(e_j,e_j)\cdot(R^{2k-2}\cdot h))(e_i,e_j,\dots,e_i,e_i)\\
&\quad -\lambda_i\lambda_j(R(e_i,e_i)\cdot(R^{2k-2}\cdot h))(e_i,e_j,\dots,e_i,e_i)\\
&\quad -\dots+2\lambda_i\lambda_j(R^{2k-1}\cdot h)(e_i,e_j,e_i,e_j,\dots,e_j,e_i)\\
&=2\lambda_i\lambda_j[(-1)^k 2^{2(k-1)}\lambda_i^{2k-1}\lambda_j^{2k-1}(\lambda_i-\lambda_j)],\\
&=(-1)^k 2^{2k-1}\lambda_i^{2k}\lambda_j^{2k}(\lambda_i-\lambda_j).
\end{aligned}$$

Hence (a) is true for k. Similarly (b) and (c) are true for k. We show that (d) is true, then (e) and (f) can be proved similarly.

$$\begin{aligned}
&(R^{2k+1}\cdot h)(e_i,e_j,e_i,e_j,\dots,e_i,e_j)\\
&\quad= -(R^{2k}\cdot h)(R(e_i,e_j)e_i,e_j,e_i,e_j,\dots,e_i,e_j)\\
&\qquad -(R^{2k}\cdot h)(e_i,R(e_i,e_j)e_j,e_i,e_j,\dots,e_i,e_j)\\
&\qquad -\dots-(R^{2k}\cdot h)(e_i,e_j,e_i,e_j,\dots,R(e_i,e_j)e_i,e_j)\\
&\qquad -(R^{2k}\cdot h)(e_i,e_j,e_i,e_j,\dots,e_i,R(e_i,e_j)e_j)\\
&\quad=\lambda_i\lambda_j[(R^{2k}\cdot h)(e_i,e_j,e_i,e_j,\dots,e_j,e_j)-(R^{2k}\cdot h)(e_i,e_j,e_i,e_j,\dots,e_i,e_i)]\\
&\quad=\lambda_i\lambda_j[(-1)^{k+1}2^{2k-1}\lambda_i^{2k}\lambda_j^{2k}(\lambda_i-\lambda_j)\\
&\qquad -(-1)^k 2^{2k-1}\lambda_i^{2k}\lambda_j^{2k}(\lambda_i-\lambda_j)\\
&\quad=(-1)^{k+1}2^{2k}\lambda_i^{2k+1}\lambda_j^{2k+1}(\lambda_i-\lambda_j).\quad\square
\end{aligned}$$

Proposition 2.1. *For a hypersurface M^n of $\mathbb{R}^{n+1}$ the following conditions are equivalent*:

(1) $R^k\cdot h=0$,

(2) $R\cdot h=0$, *i.e.* M^n *is semi-parallel,*

(3) *at each point* $p\in M^n$ *the shape operator has the following form*

$$S_p=\begin{bmatrix}\lambda & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0\end{bmatrix}.$$

Proof. If (1) holds then it follows from Lemma 2.2 that $\lambda_i\lambda_j(\lambda_i-\lambda_j)=0$. From [D] we obtain that in this case $R\cdot h=0$. Hence (2) holds. Conversely, if (2) holds, then automatically (1) holds. In other words, (1) holds if and only if $\lambda_i\lambda_j(\lambda_i-\lambda_j)=0$. From this observation follows immediately that (1) and (3) are equivalent. $\square$

In order to prove our theorem, we look at the classification of all semi-parallel hypersurfaces and select out those hypersurfaces which satisfy $\nabla^k h=0$.

§ 3. Proof of the Theorem

So let M^n be a hypersurface of $\mathbb{R}^{n+1}$ that satisfies $\nabla^k h=0$. From Lemma 2.1 and Proposition 2.1 we obtain that M^n is semi-parallel. From the classification of semi-parallel hypersurfaces then follows that there are three cases: (Case 1) M^n is an open part of a parallel hypersurface, i.e. a hyperplane, a hypersphere or a product of an affine subspace and a sphere, (Case 2) M^n is an open part of an elliptic hypercone or of the product of an elliptic cone and an affine subspace, (Case 3) M^n is flat.

Case 1. This case is trivial.

Case 2. Here it's sufficient to show that an elliptic hypercone cannot be higher order parallel. So consider the elliptic hypercone $\mathscr{C}^n=\{(x_0, x_1, \ldots x_n)\in\mathbb{R}^{n+1} \mid x_0>0$ and $(\operatorname{tg}^2\theta)\, x_0^2=x_1^2+\ldots+x_n^2\}$, where θ is a nonzero constant. From [D] it follows that $\mathscr{C}^n$ is semi-parallel and has as principal curvature functions 0 and

$$\lambda=-\frac{\cos^2\theta}{\sin\theta}\frac{1}{x_0}. \tag{3.1}$$

Note $\mathscr{C}^n$ is foliated by straight lines with parameterization $\gamma(t)=(t, a_1 t, \ldots, a_n t)$ where $(\operatorname{tg}^2\theta)=a_1^2+\ldots+a_n^2$. We consider such a line. Let X_t be a parallel unit vector field along γ that is tangent to $\mathscr{C}^n$ and orthogonal to γ'. If we denote the restriction of λ to the line γ also by λ, then $h(X_t, X_t)=\lambda(t)$, $(\nabla h)(\gamma'(t), X_t, X_t)$ $=\lambda'(t)$, and one easily sees that

$$(\nabla^k h)(\gamma'(t), \ldots, \gamma'(t), X_t, X_t)=\lambda^{(k)}(t).$$

Thus λ is a polynomial function of degree at most $k-1$ of t. But this contradicts (3.1).

Case 3. Since M^n is flat, there exists around each point of M^n an open neighbourhood U such that U is isometric to an open part of $\mathbb{R}^n$. Let $(u_1, \ldots, u_n)$ be Euclidean coordinates on U and define functions F_{ij} on U by $F_{ij}=h\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right)$. From the equation of Codazzi it follows that $\frac{\partial F_{ij}}{\partial u_k}=\frac{\partial F_{kj}}{\partial u_i}$, such that there exists a function f satisfying $F_{ij}=\frac{\partial^2 f}{\partial u_i\, \partial u_j}$. But then it follows again from the fact that M^n satisfies $\nabla^k h=0$ that all partial $(k+2)$-nd derivatives of f are zero, such that f is a polynomial function of degree at most $k+1$. Hence f can be defined on the whole of $\mathbb{R}^n$. Defining a symmetric 2-form h on $\mathbb{R}^n$ by $h\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right)$ $=\frac{\partial^2 f}{\partial u_i\, \partial u_j}$, it follows from the fundamental theorem of hypersurfaces that $\mathbb{R}^n$ can be immersed isometrically into $\mathbb{R}^{n+1}$ with h as second fundamental form and that this immersion coincides on U with our original immersion. From the

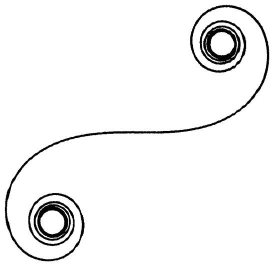

Fig. 1. $\kappa = s$

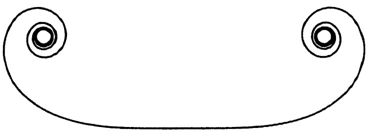

Fig. 2. $\kappa = s^2$

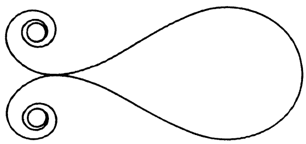

Fig. 3. $\kappa = s^2 - 2.19$

Hartman-Nirenberg cylinder theorem it then follows that U is an open part of a cylinder $\mathscr{C}^n$ over a plane curve γ.

We can suppose that γ has unit speed. After a rigid motion of $\mathbb{R}^{n+1}$, we can suppose that $\mathscr{C}^n$ has the following parameterization x:

$$x(t, u_1, \ldots, u_{n-1}) = (\gamma(t), u_1, \ldots, u_{n-1}),$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Let $\{T(t), N(t)\}$ be the oriented Frenet frame along γ and let κ be the curvature of γ. Then an orthonormal basis of the tangent space is given by $\{x_t, x_{u_1}, \ldots, x_{u_{n-1}}\}$, whereby $x_t = (T(t), 0, \ldots, 0)$ and $x_{u_j} = (0, 0, \ldots, 1, 0, \ldots, 0)$, where 1 occurs at the $(j+2)$-nd place. A unit normal vector field ξ is given by $\xi = N(t)$. Now we obtain that $\nabla_{x_t} x_t = 0$, $\nabla_{x_t} x_{u_j} = \nabla_{x_{u_j}} x_t = 0$ and $\nabla_{x_{u_j}} x_{u_k} = 0$; and also that $h(x_t, x_t) = \kappa(t)$, $h(x_t, x_{u_j}) = h(x_{u_j}, x_{u_k}) = 0$. From these formulas, it follows that $(\nabla h)(x_t, x_t, x_t) = \kappa'(t)$, and all the other first derivatives of h are zero. Now one shows easily by induction that

$$(\nabla^k h)(x_t, x_t, \ldots, x_t) = \kappa^{(k)}(t),$$

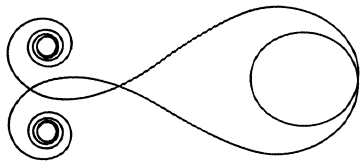

Fig. 4. $\kappa = s^2 - 4$

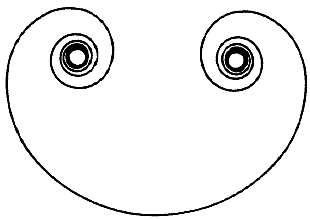

Fig. 5. $\kappa = s^2 + 1$

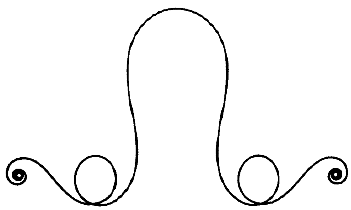

Fig. 6. $\kappa = 5s^4 - 18s^2 + 5$

and all the other k-th derivatives are zero. This shows that $\nabla^k h = 0$ if and only if the curvature function is a polynomial function of degree $k-1$ of the arc length.

Since all the hypersurfaces occurring in this local classification are analytic, they can only be pasted together if they are of the same type. This observation finishes the proof of the theorem.

§ 4. Polynomial Spirals

We call a plane curve a polynomial spiral if its curvature function is a polynomial function of the arc length. If s goes to infinity, the curvature function of course also goes to infinity, and therefore the curve spirals towards some point at its ends. Since the curvature function determines the curve up to an isometry, it is clear that every spiral γ can be written as

$$\gamma(s) = \left(\int_0^s \cos(P_k(t)), \int_0^s \sin(P_k(t)) \right). \tag{4.1}$$

Then the curvature $\kappa_\gamma(s) = P_k'(s)$ is a polynomial of degree $k-1$ of s. Thus a cylinder built on γ satisfies $\nabla^k h = 0$.

If $k=0$, then γ, defined by (4.1) is a straight line, thus a cylinder on γ is a hyperplane, which is totally geodesic, i.e. satisfying $h=0$.

If $k=1$, then γ, defined (4.1) is a circle, thus a cylinder on γ is a circular cylinder, which is parallel, i.e. satisfying $\nabla h = 0$.

If $k=2$, by measuring the arc length of γ eventually from another point, we obtain that the curvature is proportional to the arc length. All this curves are the same up to similarity transformations of the plane. Such a curve is sometimes called clothoid or more frequently the Cornu spiral, after A. Cornu who used this curve in 1874 in his study of diffraction. The Cornu spiral was probably studied first by Jacobi Bernoulli around 1696. It is described in his work OPERA, Tomus Secundus, pp. 1084–1086 [B].

Figure 1 shows how the Cornu spiral looks like. Notice that the Cornu spiral has a point of inflection at $s=0$. It doesn't have self intersections.

If $k=3$, then there are infinitely many non similar curves. By a similarity transformation and by changing the point from which we measure the arc length, we can make sure that the curvature satisfies $\kappa = s^2 - D$, where $D \in \mathbb{R}$. Figures 2, 3, 4 and 5 show how this curve looks like for $D=0$, $D=2.19$, $D=4$ and $D=-1$. Notice that γ has two points of inflection if $D>0$, one if $D=0$ and none if $D<0$. Moreover, γ has no self intersections if [D] is small.

If $k>3$, then the spirals become more and more intricate, although the general pattern remains: the curve spirals asymptotically to two points, which can coincide for some polynomials. Figure 6 gives an example of the case $k=4$.

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